Math 3151 - Substitution Examples

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1. Example.

   Integrate $\int \tan x\, dx$.

   Solution.

   $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx.$$

   The substitution

   $$u = \cos x; \quad du = -\sin x\, dx$$

   yields

   $$\begin{aligned}\int \tan x\, dx &= -\int \frac{1}{u}\, du \\ &= -\ln|u|\, du + C \\ &= -\ln|\cos x| + C.\end{aligned}$$

2. Example.

   $$\begin{aligned}\int \csc x\, dx &= \int \frac{1}{\sin x}\, dx \\ &= \int \frac{1}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\, dx \\ &= \int \frac{\sec^2\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)}\, dx.\end{aligned}$$

   The substitution

   $$u = \tan\frac{x}{2}; \quad du = \frac{1}{2}\sec^2\frac{x}{2}\, dx$$

   yields

   $$\begin{aligned}\int \csc x\, dx &= \int \frac{1}{u}\, du \\ &= \ln|u|\, du + C \\ &= \ln\left|\tan\left(\frac{x}{2}\right)\right| + C.\end{aligned}$$

3. Example.

   $$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{\cos x}\, dx \\ &= \int \frac{1}{\sin\left(x + \frac{\pi}{2}\right)}\, dx \\ &= \int \frac{1}{\sin 2\left(\frac{x}{2} + \frac{\pi}{4}\right)}\, dx \\ &= \int \frac{1}{2\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)}\, dx \\ &= \int \frac{\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}{2\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)}\, dx \\ &= \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C\end{aligned}$$

   This can be obtained by translation from the previous example.

4. Example.

   Integrate $\int x^3(x^2-1)^7\, dx$.

   Solution.

   The substitution

   $$u = x^2 - 1; \quad du = 2x\, dx$$

   yields

   $$\begin{aligned}\int x^3(x^2-1)^7\, dx &= \frac{1}{2}\int (u+1)u^7\, du \\ &= \frac{1}{2}\int u^8 + u^7\, du \\ &= \frac{1}{2}\left[\frac{u^9}{9} + \frac{u^8}{8}\right[+ C \\ &= \frac{1}{2}\left[\frac{(x^2-1)^9}{9} + \frac{(x^2-1)^8}{8}\right] + C.\end{aligned}$$

[1] http://pennance.us