The Limit of a Sequence (Brief Summary)

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1. Definition.

   A real number L is a *limit* of a sequence of real numbers if every open interval containing L contains all but a finite number of terms of the sequence.

2. Claim.

   A sequence can have at most one limit.

   Proof.

   Suppose L and L' be limits of a sequence. Then there exists an open interval I containing L and an open interval I' containing L' and both of these intervals contain all but a finite number of terms of the sequence. If $L \neq L'$, I and I' can be chosen to be disjoint which (exercise) leads to a contradiction. Thus, it must be that $L = L'$.

3. Notation.

   If L is the limit of a sequence (a_n) we write
   $$\lim_{n\to\infty} a_n = L$$
   and we say that the sequence *converges* to L or that the sequence is *convergent*. A sequence with no limit is called *divergent*.

4. Remark.

   The definition of limit gives precise meaning to the rather vague phrase *"a_n approaches L as n approaches ∞"*. This statement does not serve as the definition of limit because neither the nature of the "approach" nor the concept of ∞ are explained.

5. Lemma.

   Let I is any open interval containing L, then within I can be found an open interval of the form $|x - L| < \epsilon$

6. Theorem.

   Let (a_n) be a sequence. The following are equivalent,

   (a) $\lim\limits_{n\to\infty} a_n = L$

   (b) For every $\epsilon > 0$, all but a finite number of terms of the sequence are contained in the interval with center L and radius ϵ.

   (c) $\forall \epsilon > 0, \;\; \exists N$ such that $\forall n \in \mathbb{N}$
   $$n > N \implies |a_n - L| < \epsilon.$$

7. Example.

   Let (a_n) be the sequence given by
   $$a_n = \frac{n}{n+1}.$$
   Show that $\lim\limits_{n\to\infty} a_n = 1$.

   Solution.

   Let $\epsilon > 0$ be arbitrary. We must find a natural number N such that[2]
   $$d(a_n, 1) < \epsilon \tag{1}$$
   whenever $n > N$. Now
   $$d(a_n, 1) = \left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1}.$$
   Moreover, since $\mathbb{N}$ is unbounded above there exists a natural number N such

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[2] Notice that the value of N will depend on the value of ϵ. The smaller the value of ϵ the larger N must be in order that the open interval determined by equation (1) contains all terms after the N'th. Figures 3 and 4 suggest that that if $\epsilon = 0.2$ then $N = 5$ works and if $\epsilon = 0.1$ then $N = 9$ suffices.

that $N\epsilon > 1$. It follows that:

$$\begin{aligned} n > N &\to n > \frac{1}{\epsilon} \\ &\to \epsilon n > 1 \\ &\to \frac{1}{n} < \epsilon \\ &\to \frac{1}{n+1} < \epsilon \\ &\to d(a_n, 1) < \epsilon \end{aligned}$$

Since N has the required property, we conclude that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

8. Example.

   Let (a_n) be the constant sequence given by $a_n = 5, n \in \mathbb{N}$. Show that $\lim_{n\to\infty} a_n = 5$.

   Solution.

   Let $\epsilon > 0$. It is easy to find N such that $|a_n - 5| < \epsilon$ whenever $n > N$. Let $N = 24$ (any other natural number will do). Since $|a_n - 5| = 0$, for all $n \in \mathbb{N}$ it is clearly true that $|a_n - 5| < \epsilon$ whenever $n > 24$. It follows that $\lim_{n\to\infty} a_n = 5$.

9. More generally, if $a_n = c$ for all $n \in \mathbb{N}$ then $\lim_{n\to\infty} a_n = c$.

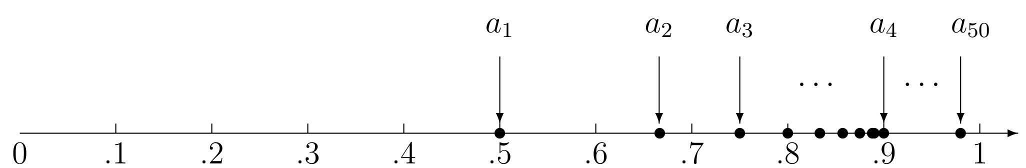


Figure 1. Initial Terms of the Sequence $a_n = \dfrac{n}{n+1}$

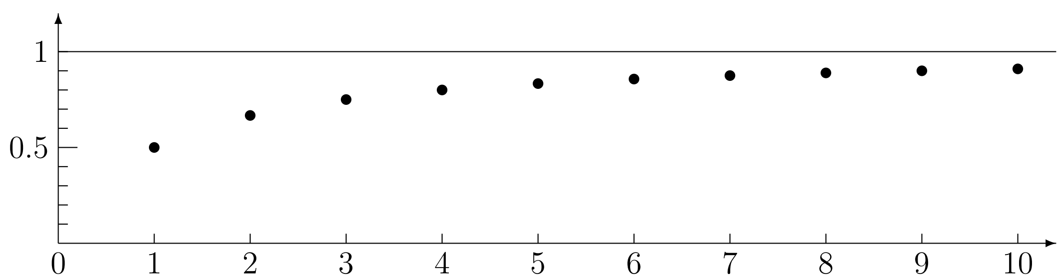


Figure 2. Graph of the sequence $a_n = \dfrac{n}{n+1}$

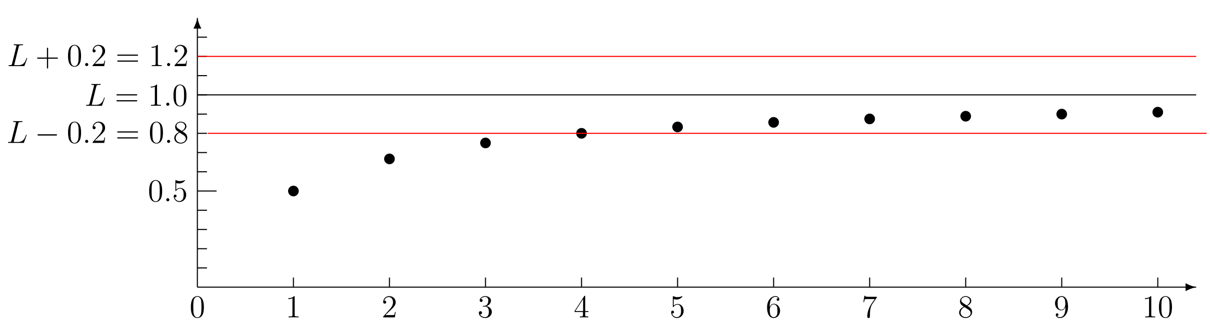


Figure 3. $n > 5 \to |a_n - 1| < 0.2$.

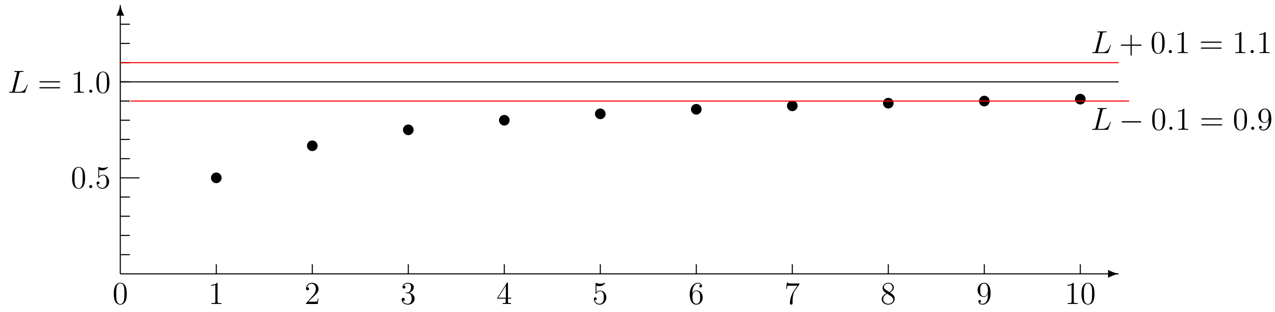


Figure 4. $n > 9 \to |a_n - 1| < 0.1$.

10. Example.

Let (a_n) be the sequence given by

$$a_n = \frac{1}{n-1}, \quad n > 1.$$

Show using the definition of the limit of a sequence that

$$\lim_{n\to\infty} a_n = 0.$$

Solution.

Let $\epsilon > 0$ Notice (exercise) that $\frac{1}{n-1} < \frac{2}{n}$ for all $n > 2$. Since $\mathbb{N}$ is unbounded, there exists a natural number $N > 1$ such that $N\epsilon > 2$. It follows that:

$$\begin{aligned} n > N &\to n\epsilon > 2 \\ &\to \frac{2}{n} < \epsilon \end{aligned}$$

Since $\dfrac{1}{n-1} < \dfrac{2}{n}$, it follows that

$$\begin{aligned} n > N &\to \frac{1}{n-1} < \epsilon \\ &\to \left| \frac{1}{n-1} - 0 \right| < \epsilon \\ &\to d(a_n, 0) < \epsilon \end{aligned}$$

which proves that

$$\lim_{n\to\infty} \frac{1}{n-1} = 1$$

11. The next theorem reduces the calculation of complex limits to simpler ones.

12. Theorem.

Let (a_n) and (b_n) be sequences. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then:

(a) $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

(b) $\lim_{n\to\infty} (a_n \cdot b_n) = L \cdot M$

(c) If $M \neq 0$ and $b_n \neq 0$ for all n then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}.$$

13. Example.

Find the limit of the sequence (a_n) given by

$$a_n = 3/n$$

Solution.

Notice $a_n = x_n \cdot y_n$ where $x_n = 1/n$ and $y_n = 3, \quad n \in \mathbb{N}$. Now, it is easy to show that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 3$. Using part (b) of the above theorem yields

$$\lim_{n\to\infty} a_n = 0 \cdot 3 = 0$$

14. Example.

Find the limit of the sequence (a_n) given by

$$a_n = \frac{1}{n^2}.$$

Solution.

Let $x_n = 1/n$. Then

$$a_n = x_n \cdot x_n$$

Since $\lim_{n\to\infty} x_n = 0$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n \\ &= 0 \cdot 0 \\ &= 0.\end{aligned}$$

15. Using mathematical induction (exercise) it is now easy to show that

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

for any natural number $p > 0$.

16. Example.

Find the limit of the sequence

$$x_n = \frac{n^2 + 3n}{3 + n^2}.$$

Solution.

$$\begin{aligned}x_n &= \frac{\frac{n^2}{n^2} + \frac{3n}{n^2}}{\frac{3}{n^2} + \frac{n^2}{n^2}} \\ &= \frac{1 + \frac{3}{n}}{\frac{3}{n^2} + 1}\end{aligned}$$

But

$$\begin{aligned}\lim_{n\to\infty} \left(1 + \frac{3}{n}\right) &= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{3}{n} \\ &= 1 + 0 \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{3}{n^2} + 1\right) &= \lim_{n\to\infty} \frac{3}{n^2} + \lim_{n\to\infty} 1 \\ &= 0 + 1 \\ &= 1\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty} x_n &= \frac{\lim_{n\to\infty} \left(1 + \frac{3}{n}\right)}{\lim_{n\to\infty} \left(\frac{3}{n^2} + 1\right)}. \\ &= 1.\end{aligned}$$

17. Definition.

A sequence (a_n) is *bounded* if there exists a number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. A sequence which is not bounded is called *unbounded*.

18. Example.

Consider the sequence $a_n = \dfrac{n}{n+1}$. Since $|a_n| \leq 1$ for all $n \in \mathbb{N}$, this sequence is bounded.

19. Claim.

A convergent sequence is bounded.

Proof. Let (a_n) be a convergent sequence with limit L. Then there exists a number N such the open interval $(L-1, L+1)$ contains all terms a_n with $n > N$. It follows that all terms of a_n are less than or equal in absolute value to the maximum M of the set

$$\{|a_1|, |a_2|, \ldots, |a_N|, |L+1|, |L-1|\}$$

(Exercise: Find a counterexample to show that the converse of this result is false.)

20. Corollary.

The sequence $a_n = n$ has no limit.

Proof. Notice that the sequence (a_n) is unbounded. If it had a limit it would be bounded.

21. Example.

Consider the sequence (a_n) where

$$a_n = 3^n.$$

Then

$$\begin{aligned} 3^n &= (1+2)^n \\ &\geq 1+2n \\ &\geq n \end{aligned}$$

It follows that the sequence (a_n) is unbounded and hence has no limit.

22. Sandwich Theorem.

 Let (a_n), (b_n) and (c_n) be sequences. If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and the seqences (a_n) and (c_n) have the same limit L then $\lim_{n\to\infty} b_n = L$

23. Example.

 Show $\lim\limits_{n\to\infty} \left|\frac{\cos n}{n}\right| = 0$

 Solution.

 Notice that for all $n \geq 1$

$$0 \leq \left|\frac{\cos n}{n}\right| \leq \frac{1}{n}$$

 The result follows by applying the sandwich theorem.

24. Example.

 Show that $\lim\limits_{n\to\infty} \frac{n^4}{5^n} = 0$.

 Solution.

 By the binomial theorem

$$\begin{aligned} 5^n &= (1+4)^n \\ &\geq \binom{n}{5} 4^5 \end{aligned}$$

 Let $p(n) = \binom{n}{5} 4^5$. Then

$$0 \leq \frac{n^4}{5^n} \leq \frac{n^4}{p(n)}$$

 Since p is a polynomial of degree 5, the right hand side convegres to 0 and the result then follows from the sandwich theorem.

25. Claim.

 The following are equivalent:

 (a) $\lim\limits_{n\to\infty} a_n = 0$.

 (b) $\lim\limits_{n\to\infty} |a_n| = 0$.

 Proof. Exercise

 Hint: Notice that

$$d(|a_n|, 0) = d(a_n, 0)$$

26. Corollary.

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

27. Lemma (Bernoull's Inequality).

 If $x > -1$ then

$$(1+x)^n \geq (1+nx)$$

 Proof. By induction. (Exercise).

28. Claim.

 If $a > 1$ then a^n is unbounded above.

 Proof. If $a > 1$ then $a = 1 + x$ where $x = a - 1$. Notice $x > 0$. By Bernoulli's inequality

$$a^n = (1+x)^n > 1 + nx$$

 Since $\mathbb{N}$ is unbounded above, given M arbitrary, there exists $N \in \mathbb{N}$ such that for all $n > N$, $nx > M - 1$. Hence, for all $n > N$ $a^n > M$ proving that the sequence (a^n) is unbounded above.

29. Claim.

 If $0 < |x| < 1$ then

$$\lim_{n\to\infty} |x|^n = 0 \tag{2}$$

 Proof.

 If $0 < |x| < 1$, there exists $a > 0$ such that

$$|x| = \frac{1}{1+a}$$

Hence, using Bernoulli's inequality

$$0 \le |x|^n = \frac{1}{(1+a)^n} \le \frac{1}{1+na}.$$

Equation (2) now follows from the sandwich theorem.

30. Corollary.
If $|x| < 1$ then

$$\lim_{n\to\infty} x^n = 0$$

31. Example.
Show that the sequence $a_n = 3^{1/n}$ converges.
Solution.

By Bernoulli's inequality

$$\left(1+\frac{2}{n}\right)^n \ge 1 + n\cdot\frac{2}{n} = 3$$

and so

$$3^{1/n} \le 1 + \frac{2}{n}$$

Again, by Bernoulli's inequality,

$$\left(1-\frac{2}{3n}\right)^n \ge 1 - n\cdot\frac{2}{3n} = \frac{1}{3}$$

and so

$$3^{1/n} \ge \frac{1}{1-\dfrac{2}{3n}}$$

It follows by the sandwich theorem that

$$\lim_{n\to\infty} 3^{1/n} = 1.$$

32. Claim.
If $a > 0$ then $\lim_{n\to\infty} a^{1/n} = 1$.
Proof. (Exercise)
(Hint: If $1 < a$ see the previous example. If $0 < a < 1$ consider the reciprocal).

33. Claim.

$$\lim_{n\to\infty} n^{1/n} = 1. \qquad (3)$$

Proof.
Notice (exercise) that $n^{1/n} > 1$. Hence, using the binomial theorem,

$$\begin{aligned} n &= \left[n^{1/n}\right]^n \\ &= \left[1 + (n^{1/n}-1)\right]^n \\ &\ge \binom{n}{2}(n^{1/n}-1) \end{aligned}$$

Therefore,

$$0 \le (n^{1/n}-1) \le \sqrt{\frac{2}{n-1}}$$

and (3) now follows from the sandwich theorem.

34. Definition.
A sequence (x_n) in
R is:

(a) *increasing* if for all $n, m \in \mathbb{N}$

$$n < m \to x_n < x_m$$

(b) *non-decreasing* if for all $n, m \in \mathbb{N}$

$$n \le m \to x_n \le x_m$$

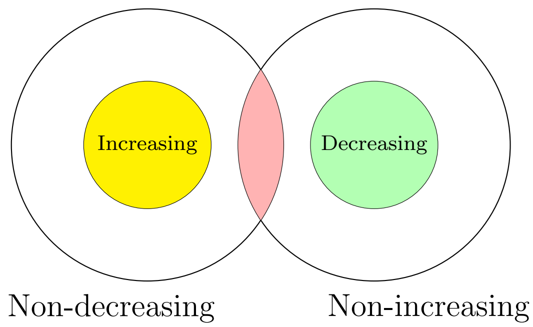
(c) *decreasing* if for all $n, m \in \mathbb{N}$

$$n < m \to x_n > x_m$$

(d) *non-increasing* if for all $n, m \in \mathbb{N}$

$$n \le m \to x_n \ge x_m$$

The following Venn diagram, expresses the relationships between these concepts:



35. Dedekind (Completeness) Axiom.

Every non-empty subset of R which is bounded above has a supremum (least upper bound).

36. Claim.

A sequence $\alpha = (a_n)$ of real numbers which is both non-decreasing bounded converges to the least upper bound of its set of terms.

Proof. Let $L = \sup\{a_n : n \in \mathbb{N}\}$ and $\epsilon > 0$. Then there exists N such that

$$L - \epsilon < a_N \leq L$$

(otherwise $L - \epsilon$ would be a smaller bound). Since α is non-decreasing

$$n \geq N \to L - \epsilon < a_n \leq L$$

Hence α converges to L.

37. Corollary.

A non-decreasing sequence $\alpha = (a_n)$ of real numbers either converges or is unbounded above.

38. Definition.

Let (x_n) be a real sequence. A natural number m is called:

(a) A *peak point* of (x_n) if $x_m \geq x_n$ for all $n \geq m$.

(b) A *p-blocker* if $m > p$ and $x_m \geq x_p$

39. Bolzano Weierstrass Theorem [BW].

A bounded sequence has a convergent subsequence.

Proof. (See Spivak).

Let $m_1 < m_2, < \ldots,$ be the set of peak points. If this set is infinite then the bounded non increasing subsequence

$$x_{m_1} \geq x_{m_2} \geq x_{m_k} \ldots$$

must converge (to its infimum). If, on the other hand, the set of peak points is finite, there exists a last peak point m in which case there exists a convergent subsequence of m-blockers defined recursively as follows: let $q_1 = m + 1$ and, for each k, let q_{k+1} be the minimum blocker of q_k. Then the subsequence

$$x_{q_1} \leq x_{q_2} \leq x_{q_k} \ldots$$

is non decreasing and bounded below so converges.

40. Exercise.

In the proof of BW above, explain why the minimum at each stage exists.

41. Definition.

A real sequence (a_n) is *Cauchy* if for every $\epsilon > 0$ there exists N such that

$$n, m > N \to |a_n - a_m| < \epsilon$$

.

42. Claim.

Cauchy sequences are bounded.

Proof. Let $\epsilon > 0$. There exists N such that for $m, n \geq N$, $|a_m - a_n| < \epsilon$. Then, by the triangle inequality

$$\begin{aligned} |a_m| - |a_n| &\leq |a_m - a_n| \\ &< \epsilon \end{aligned}$$

for all $m, n \geq N$. Taking $n = N$. Then

$$|a_m| - |a_N| < \epsilon$$

for all $m \geq N$. It follows easily that (a_m) is bounded by

$$\pm \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|, |a_N|, \epsilon + |a_N|\}$$

43. Claim [Uses BW]. Every Cauchy sequence in
R converges.

Proof. Let (a_n) be a Cauchy sequence of real numbers. Then (a_n) is bounded so by BW has a convergent subsequence (a_{n_k}). Let L be the limit of this subsequence. Then there exist N such that

$$n_k > N \to |a_{n_k} - L| < \epsilon/2$$

and there exists M such that

$$n, n_k > M \to |a_n - a_{n_k}| < \epsilon/2$$

Then, for all $n > \max\{N, M\}$

$$\begin{aligned} |a_n - L| &\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\ &< \epsilon \end{aligned}$$

44. Corollary.

Dedekind completeness implies Cauchy completeness