First Mean Value Theorem for Integrals

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1. Definition.

   Let f be continuous on $[a, b]$. The mean of f on $[a, b]$ is the quantity

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

2. Claim. (First Mean Value Theorem for Integrals.)

   Let f be continuous on $[a, b]$. Then there exists $\xi \in [a, b]$ such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

   Proof.

   Since f is continuous, it achieves its maximum M and minimum m on $[a, b]$. Hence, if $h(x) = (b-a)f(x)$, then

$$m(b-a) \le h(x) \le M(b-a)$$

   Notice that $m(b-a)$ is a lower sum of the function f and $M(b-a)$ is an upper sum. Since the integral of a function lies between every lower sum and every upper sum:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

   Since h is continuous on $[a, b]$ it also takes every value between its maximum and minimum. In particular, there exists $\xi \in [a, b]$ such that

$$h(\xi) = \int_a^b f(x)\,dx$$

   It follows that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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