Math 3151 -Exercises on the Definition of the Integral

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1. (a) Define the concepts of definite integral and indefinite integral.
   (b) Let $f(x) = x + 1$ and let P be the partition $\{1, 3, 5, 7\}$ of the interval $[1, 10]$
   (c) Find the lower sum $L(f, P)$.
   (d) Find the upper sum $U(f, P)$.
2. Consider the function $f(x) = |x|$ and the partition
$$P = \{-1, -3/4, -1/4, 0, 1/4, 3/4, 1\}$$
of the interval $[-1, 1]$. Find and simplify the lower sum $L(f, P)$ and upper sum $U(f, P)$ of f for the partition P.
3. Let $f(x) = x + 1, \quad 0 \le x \le 1$. Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 1]$ such that
$$x_i - x_{i-1} = \frac{1}{n}, \quad i = 1, 2, ..., n.$$
   (a) Find the lower sum $L(f, P_n)$.
   (b) Find the upper sum $U(f, P_n)$.
   (c) Find the limit of the lower sum as $n \to \infty$.
   (d) Show that that
$$U(f, P) - L(f, P)) = \frac{f(1) - f(0)}{n}.$$
4. Let f be a function which is continuous and increasing in the interval $[0, 1]$. Let $P = \{x_0, x_1, \ldots, x_n\}, \quad n \in \mathbb{N}$ be a partition f the interval such that $x_i - x_{i-1} = 1/n, \quad i = 1, 2, ..., n$. Verify that
$$U(f, P) - L(f, P)) = \frac{f(1) - f(0)}{n}.$$
5. Compute $\displaystyle\int_1^5 f(x)dx$ using the definition of the definite integral.
6. Let $f(x) = x^2$ and $0 \le a < b$. Let
$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$
be any partition of the interval $[a, b]$. Show that
$$U(f, P) \ge b^3/3 - a^3/3.$$
7. Let $f(x) = x$ and $0 \le a \le b$. Let
$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$
be any partition of the interval $[a, b]$. Show that
$$L(f, P) \le \frac{b^2}{2} - \frac{a^2}{2}.$$
8. Let $f(x) = 1/x, \quad 1 \le x \le 1 + \frac{1}{n}$. Let $P = \{1, 1 + \frac{1}{n}\}$ be be the partition of the interval $[1, 1 + \frac{1}{n}]$ consisting of only two endpoints .
   (a) Compute the lower sum $L(f, P)$. and the upper sum $U(f, P)$.
   (b) Show that
$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$
   (c) Use the result of (c) to show that
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

[1] https://pennance.us