Exponential growth and decay

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1. A population of bacteria doubles every hour[2]. Let $\phi(t)$ be the population at time t. Let $x_0 = \phi(t_0)$ be the initial population at some time t_0.

   (a) Show, by induction, that the population n hours later is given by:

   $$\phi(t_0 + n) = x_0 2^n, \quad n \in \mathbb{N}$$

   (b) Find the value of the constitutive constant k such that at time $t = t_0 + n$ the population is given by the equation

   $$\phi(t) = x_0 e^{k(t-t_0)}, \quad n \in \mathbb{N}$$

   (c) Assume that the population can be modeled by replacing the natural number n by a continuous variable t so that for $t \in \mathbb{R}$ with $t \geq t_0$

   $$\phi(t) = x_0 e^{k(t-t_0)}$$

   Verify that the function ϕ is a solution of the initial value problem:

   $$x' = kx, \quad x(t_0) = x_0$$

   (d) Draw the slope field and a family of solution curves for the equation $x' = kx$.

   (e) Consider the solution:

   $$\phi(t) = x_0 e^{k(t-t_0)}, \quad t \in \mathbb{R}$$

   of the initial value problem

   $$x' = kx, \quad x(t_0) = x_0$$

   Show that any other solution

   $$\psi : I \to \mathbb{R},$$

   where I is an interval, is a restriction of ϕ. Hint: Consider the derivative of the function $\psi(t)e^{-k(t-t_0)}$.

2. For a chemical reaction

   $$\mathrm{aA} + \mathrm{bB} \longrightarrow \mathrm{mM} + \mathrm{nN}$$

   occurring in a closed system:

   $$\frac{1}{m}\frac{d[M]}{dt} = \frac{1}{n}\frac{d[N]}{dt} = -\frac{1}{a}\frac{d[A]}{dt} = \frac{1}{b}\frac{d[B]}{dt}$$

   where [X] denote the concentration of species X. The common value r is called the *rate constant.* For example, if $2\mathrm{NO(g)} + \mathrm{O_2(g)} \longrightarrow 2\mathrm{NO_2(g)}$ then:

   $$r = \frac{1}{2}\frac{d[NO_2]}{dt} = -\frac{d[O_2]}{dt} = -\frac{1}{2}\frac{d[NO]}{dt}$$

   A negative sign informs us that the corresponding species is a reactant. For many reactions a rate law of the form $r = k[A]^n[B]^m \ldots [Z]^p$ holds. In this case, the sum of the exponents is called the *order* of the reaction.

   For a reaction of the form $\mathrm{A} \longrightarrow \mathrm{mM} + \mathrm{nN}$, let $[\mathrm{A}](t)$ be the concentration of reactant A at time t. Let $[A]_0$ denote the initial concentration at time t_0.

   (a) If $[A]$ satisfies the zero order reaction equation $-\dfrac{d[A]}{dt} = k$ show that

   $$[A](t) = [A]_0 - k(t - t_0).$$

   (i.e. the concentration of reactant A decreases linearly from its initial value of $[A]_0$.

[1] https://pennance.us

[2] Remark: In reality populations never grow to infinity. Hence exponential increase is not a good model for estimating human population (which are actually declining at an alarming rate). However, following Malthus, it is widely used for its propaganda value by extreme environmentalists and climate change alarmists and, worse, is widely believed.

(b) If $[A]$ satisfies the first order reaction equation $\frac{d[A]}{dt} = -k[A]$ show that:

$$[A](t) = [A]_0 e^{-k(t-t_0)}.$$

(c) If $[A]$ satisfies the second order reaction equation $\frac{d[A]}{dt} = -k[A]^2$ show that

$$[A](t) = \frac{[A]_0}{1 + [A]_0 k(t - t_0)}.$$

3. The charge on a capacitor as a function of time is $Q(t) = 2e^{-t}, \quad t \geq 0$

   (a) Find the average rate of change of Q in the time interval $[0, 10]$.

   (b) Find the instantaneous rate of change of Q as a function of time.

   (c) Find the time τ at which the the quantity of charge is equal to the average charge in the interval $[0, 10]$.

4. The *half-life* $t_{1/2}$ of a reactant A is the time required for $[A]$ to drop from its initial value $[A]_0$ to half the initial value. Show that for a first order reaction: $A + \cdots \longrightarrow$ Products, the half life is given by: $t_{1/2} = \frac{\ln 2}{k}$.

5. The *half-life* of the plutonium isotope Pu-239 is $T = 24,100$ years.

   (a) Determine the time τ it will take for the 30 grams to decay to 5 grams.

   (b) Express the time τ as a function of T.

6. Let $\theta(t)$ be the temperature of a cooling body at time t. According to Newton's law of cooling, θ satisfies the ordinary differential equation. $\theta' = -k(\theta - \hat{\theta})$ where $k > 0$ is a constitutive constant and $\hat{\theta}$ is the ambient temperature -assumed constant. Find the general solution of this equation. Sketch the direction field and typical solutions.