Validity and Proof

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1. Aristotle (384-322 BC) studied arguments which consisted of two or more statements called *premises* and a statement called the *conclusion*. We will be interested mainly in the special case of arguments with two premises F and G and one conclusion C. Such arguments are often presented in the format:

$$\begin{array}{c} F \\ G \\ \hline C \end{array}$$

2. Definition.

   An argument with two premises F and G and a conclusion C is called *valid* if the truth of the conjunction of the premises $F \wedge G$ is a sufficient condition for the truth of the conclusion C (i.e., if the implication $F \wedge G \Rightarrow C$ is true).

3. Remark.

   To check the validity of an argument, it suffices to show that the conclusion is true whenever the conjunction of the premises is true.

4. Example.

   Consider an argument of the form:

$$\begin{array}{l} F \\ F \Rightarrow G \\ \hline G \end{array}$$

   The two premises of this argument are F and $F \Rightarrow G$ and the conclusion is G. To show that the argument is valid, suppose that both premises are true (i.e. suppose that F is true and that the implication $F \Rightarrow G$ is also true). The truth of the latter implication means that it CANNOT happen that F be true and at the same time G be false. Since F is true by hypothesis, it can only be that G is true. Thus, the conclusion of the argument is true whenever both premises are true. Therefore, the argument is valid.

5. Definition.

   The form of arguemt in item (**??**) is called *modus ponens*.

6. Validity of an argument does not depend on the nature of the statements involved but only the form of the argument. Any statements which we substitute for F and G in the above form will produce a valid argument.

7. Example. Let F be the statement, "Rex is a dog", and G the statement, "Rex has fleas" then the argument

$$\begin{array}{l} \text{Rex is a dog} \\ \text{If Rex is a dog then Rex has fleas} \\ \hline \text{Rex has fleas} \end{array}$$

   has the form modus ponens which we already have shown to be valid.

   Notice that that the validity of an argument form does not depend on the truth or falsehood of the premises. The argument in the previous example is valid even if the premise Rex is a dog is false.

8. Definition.

   An argument is called *sound* if It is valid and the premises are true.

9. Example.

   The argument

$$\begin{array}{l} \text{If } n > 2 \text{ then } n^2 > 4 \\ n^2 > 4 \\ \hline n > 2 \end{array}$$

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has the form

$$\begin{array}{l} F \Rightarrow G \\ G \\ \hline F \end{array}$$

An argument of this form is invalid since both premises can be true and at the same time the conclusion F false. This is the case, for example, when F is false and G is true. In this case $n = -3$ provides a counter example.

10. Example.

Show that the argument

$$\begin{array}{l} \neg p \\ p \rightarrow q \\ \hline \neg q \end{array}$$

is invalid

Solution.

Recall that an argument is *valid* if it cannot happen that all the premises are true and the conclusion false. This argument is invalid because if p is false and q true then $p \rightarrow q$ is true and $\neg p$ is true. Hence both premises are true yet the conclusion $\neg q$ is false.

11. Example.

Decide the validity of

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow p \\ \hline p \leftrightarrow q \end{array}$$

Solution. The argument is valid. To see this, suppose both premises are true. From the truth of the premise $p \rightarrow q$ it follows that it cannot happen that p is true and q false. From the truth of the premise $q \rightarrow p$ it follows that it cannot happen that q is true and p false. It can only be that p and q have the same truth values. This means that the conclusion $p \leftrightarrow q$ is true.

12. Example.

Explain why the argument form

$$\begin{array}{c} \neg(p \wedge q) \\ \hline \neg p \vee \neg q \end{array}$$

is valid.

Solution.

Validity of this argument follows from immediately from the linguistic definitions of conjunction (AND) and disjunction (OR). The fact that the negation of a conjunction is the disjunction of the negations is a property of the language. If it is NOT true that I eat both in Burger King AND MacDonald's then either I do NOT eat in Burger King OR I do NOT eat in MacDonald's.

13. Example.

Show that the argument form

$$\begin{array}{c} p \rightarrow q \\ \neg p \rightarrow \neg q \\ \hline p \leftrightarrow q \end{array}$$

is valid.

Solution.

Suppose both premises are true. From the truth of $p \rightarrow q$ it follows that q is true whenever p is true. From the truth of $\neg p \rightarrow \neg q$ it follows that q is false whenever p is false. Hence p and q must have the same truth values. Therefore the conclusion $p \leftrightarrow q$ must also be true.

14. Example.

Explain why the argument form

$$\begin{array}{c} \neg(p \wedge \neg q) \\ \hline p \rightarrow q \end{array}$$

Solution.

Notice that the premise $\neg(p \wedge \neg q)$ is merely a way of expressing implication in terms of negation (¬) and conjunction (∧). If the premise is true, it CANNOT happen that p is true AND, at the same time, the consequent q false. By the definition of implication, this means that $p \to q$ is true.

15. Definition.

    A *proof* of a mathematical statement F is a finite sequence of statements $F_1, F_2, \dots F_n$ such that

    (a) $F_n = F$

    (b) Each statement F_i is either an axiom or follows from the preceeding statements by valid arguments.

16. Definition.

    A statement of the form $G \wedge \neg G$ is called a *contradiction.*

    The valid argument

    $$\frac{F \Rightarrow (G \wedge \neg G)}{\neg F}$$

    is called *proof by contradiction.*

    Remark. The idea is that if a statement F implies a contradiction, then the negation of that statement must be true.

17. Example.

    Use proof by contradiction to show that there do not exist natural numbers n and m such that

    $$2m + 16n = 21 \qquad (1)$$

    Solution.

    Suppose to the contrary that there exist natural numbers n and m satisfying (**??**). Since the left hand side of (**??**) is even and hence not odd, we have the contradiction "21 is not odd and 21 is not odd" It follows (from the validity of proof by contradiction) that n and m do not exist.

Exercises

1. Show that each of the following argument forms are valid.

   (a) $\dfrac{\neg(F \vee G}{\neg F \wedge \neg G}$ (b) $\dfrac{\neg G \Rightarrow \neg F}{F \Rightarrow G}$ (c) $\dfrac{\begin{array}{l}F \Rightarrow G \\ G \Rightarrow H\end{array}}{F \Rightarrow H}$ (d) $\dfrac{F \Rightarrow (G \wedge \neg G)}{\neg F}$

2. Write the following arguments in symbolic form and check for validity.

   (a) If it rains I use an umbrella
I use an umbrella
———
It is raining

   (b) If it rains I use an umbrella
I do not use an umbrella
———
It is not raining

   (c) If it rains I use an umbrella
It is not raining
———
I am not using an umbrella

3. Which of the following arguments are valid? Prove your answers.

   (a) $\dfrac{\begin{array}{l}F \Rightarrow G \\ \neg G\end{array}}{\neg F}$ (b) $\dfrac{\begin{array}{l}F \\ G \Rightarrow F\end{array}}{F}$ (c) $\dfrac{\begin{array}{l}\neg F \\ F \Rightarrow G\end{array}}{\neg G}$ (d) $\dfrac{\begin{array}{l}\neg F \\ F \Rightarrow G\end{array}}{G}$

4. Consider the following argument.

    Everything which begins to exist
    has a cause.
    The universe began to exist.

    The universe has a cause.

    (a) Is the argument valid? Justify your answer.

    (b) Is the argument solid? Discuss.

5. Consider the following argument:

    If God existed there would be no war.
    There is war.

    God does not exist.

    (a) Is the argument valid? Justify your answer.

    (b) Is the argument solid? Discuss.

6. Show that the argument form *modus ponens* is valid if and only if the formula

$$(p \wedge (p \Rightarrow q)) \Rightarrow q$$

    is a tautology.

7. Prove the following statements by contradiction:

    (a) The negative of any even integer is even.

    (b) For all integers n, m, if $n - m$ is even then $n^3 - m^3$ is even.

    (c) The sum of any two rational numbers is rational.

    (d) There do not exist natural numbers n, m such that $14n + 20m = 101$.

    (e) If n^2 is even then n is even.

    (f) There do not exist integers a, b with $a^2/b^2 = 2$.

    (g) There do not exist integers a, b with $a^2/b^2 = 9/2$.

    (h) $3/\sqrt{2} \notin \mathbb{Q}$.

    (i) Let n be an integer. If 9 divides n then so does 3.

    (j) Let $x \in \mathbb{R}$. Then, either $\sqrt{3} + x \notin \mathbb{Q}$ or $\sqrt{3} - x \notin \mathbb{Q}$

Appendix - Useful Valid Arguments

1. **Double Negation**
$$\begin{array}{l} \neg\neg p \\ \hline p \end{array}$$

2. **Modus Ponens**
$$\begin{array}{l} p \to q \\ p \\ \hline q \end{array}$$

3. **Modus Tollens**
$$\begin{array}{l} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

4. **Absurdity (Contradiction)**
$$\begin{array}{l} p \to (q \wedge \neg q) \\ \hline \neg p \end{array}$$

5. **Addition Laws**

   (a) $$\begin{array}{c} p \\ \hline p \vee q \end{array}$$

   (b) $$\begin{array}{c} p \\ \hline p \wedge (q \vee \neg q) \end{array}$$

6. **Simplification Laws**

   (a) $$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

   (b) $$\begin{array}{l} p \wedge (q \vee \neg q) \\ \hline p \end{array}$$

7. **De Morgan's Laws**

   (a) $$\begin{array}{l} \neg(p \wedge q) \\ \hline \neg p \vee \neg q \end{array}$$

   (b) $$\begin{array}{l} \neg(p \vee q) \\ \hline \neg p \wedge \neg q \end{array}$$

8. **Contraposition**
$$\begin{array}{l} \neg q \to \neg p \\ \hline p \to q \end{array}$$

9. **Syllogism**
$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

10. **More Arguments Involving Implication**

   (a) $$\begin{array}{l} \neg(p \wedge \neg q) \\ \hline p \to q \end{array}$$

   (b) $$\begin{array}{l} p \to q \\ q \to p \\ \hline p \leftrightarrow q \end{array}$$

   (c) $$\begin{array}{l} p \to q \\ \neg p \to \neg q \\ \hline p \leftrightarrow q \end{array}$$

11. **Distributive Laws**

   (a) $$\begin{array}{c} p \wedge (q \vee r) \\ \hline (p \wedge q) \vee (p \wedge r) \end{array}$$

   (b) $$\begin{array}{c} p \vee (q \wedge r) \\ \hline (p \vee q) \wedge (p \vee r) \end{array}$$

12. **Commutative Laws**

   (a) $$\begin{array}{l} p \vee q \\ \hline q \vee p \end{array}$$

   (b) $$\begin{array}{l} p \wedge q \\ \hline q \wedge p \end{array}$$

13. **Associative Laws**

   (a) $$\begin{array}{l} (p \vee q) \vee r \\ \hline p \vee (q \vee r) \end{array}$$

   (b) $$\begin{array}{l} (p \wedge q) \wedge r \\ \hline p \wedge (q \wedge r) \end{array}$$