Trigonometric Addition Formulae

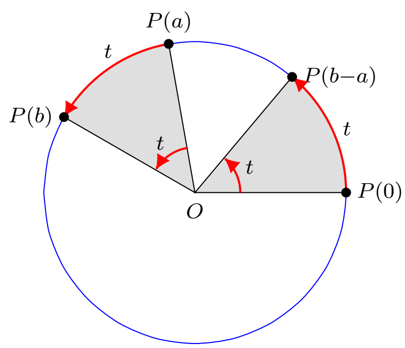
Philip Pennance[1] –Version: March 2021.

1. Claim. (First addition formula)

   Let $a, b \in \mathbb{R}$.

   $$\cos(b-a) = \cos b \cos a + \sin b \sin a \quad (1)$$

   Proof.

   Consider the trigonometric points

   $$P(b) = (\cos b, \sin b)$$
   $$P(a) = (\cos a, \sin a)$$

   and let $t = b - a$. By definition of P, the sector $P(a)OP(b)$ of the unit circle has angle $\angle P(a)OP(b) = t$. The sector $P(0)OP(b-a)$ has the same angle t and so the two sectors are congruent. Hence

   $$d[P(b), P(a)] = d[P(b-a), P(0)].$$

   Since $P(b-a) = (\cos(b-a), \sin(b-a))$. It follows from the distance formula that $d[P(b-a), P(0)]$ equals

   $$\sqrt{(\cos(b-a)-1)^2 + \sin^2(a-b)}.$$

   Also, by the distance formula $d[P(b), P(a)]$ equals

   $$\sqrt{(\cos b - \cos a)^2 + (\sin b - \sin a)^2}.$$

   Formula (1) now follows (exercise) by equating these two expressions and recalling the Pythagorean formula $\cos^2 x + \sin^2 x = 1$.

   Remark. All of the following results follow immediately from the first addition formula (1) .

2. Claim. (Cosine is even.)

   $$\cos(-a) = \cos a, \quad \forall a \in \mathbb{R}.$$

   Proof.

   Put $b = 0$ in (1).

3. Claim. (Rotation by π)

   $$\cos(b \pm \pi) = -\cos b \quad (2)$$

   Proof.

   Put $a = \pm\pi$ in (1).

4. Claim. (Rotation by $\pi/2$)

   $$\cos\left(b - \frac{\pi}{2}\right) = \sin b$$
   $$\sin\left(b + \frac{\pi}{2}\right) = \cos b$$

   Proof.
   The first equation follows by putting $a = \frac{\pi}{2}$ in (1). The second equation is a corollary of the first.

   Remark. This shows that the graphs of sin and cos are each translations of each other.

5. Claim (Second addition formula).

   $$\cos(b+a) = \cos b \cos a - \sin b \sin a$$

   Proof.

   Replace a by $-a$ in (1) and recall that cos is even and sin odd.

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6. Claim (Third addition formula).

$$\sin(b+a) = \sin b \cos a + \cos b \sin a$$

Proof.

Using the second addition formula and the rotation formulae,

$$\begin{aligned}\sin(b+a) &= \cos\left(b+a-\tfrac{\pi}{2}\right)\\ &= \cos\left(b+\left(a-\tfrac{\pi}{2}\right)\right)\\ &= \cos b \cos\left(a-\tfrac{\pi}{2}\right) - \sin b \sin\left(a-\tfrac{\pi}{2}\right)\\ &= \cos b \sin a - \sin s \cos(a-\pi)\\ &= \cos b \sin a + \sin b \cos a.\end{aligned}$$

7. Claim (Fourth addition formula).

$$\sin(b-a) = \sin b \cos a - \cos b \sin a.$$

Proof.

Replace a by $-a$ in the third addition formula.

8. Corollary (Rotation by π.)

$$\sin(b \pm \pi) = -\sin b.$$

9. Claim. (Double Angle Formulae)

$$\cos 2a = \cos^2 a - \sin^2 a \tag{3}$$
$$\cos 2a = 1 - 2\sin^2 a \tag{4}$$
$$\cos 2a = 2\sin^2 a - 1. \tag{5}$$

Proof.

(3) follows by putting $b = a$ in the second addition formula. (4) and (5) follow from the first using the pythagorean formula.

10. Claim. (Double Angle Formulae)

$$\sin 2a = 2 \sin a \cos a.$$

Proof. (Exercise.)

11. Remark. In conjuntion with the pythagorean identity

$$\cos^2 x + \sin^2 x = 1$$

the double angle formula (4) can be used to evaluate the sin and cos of a half angle.

12. Example.

Find the exact value of $\cos \frac{\pi}{12}$.

Solution.

Notice that $\frac{\pi}{12} = \frac{1}{2} \cdot \frac{\pi}{6}$. The pythagorean identity and (3) respectively give

$$\begin{aligned}\cos^2 \tfrac{\pi}{12} + \sin^2 \tfrac{\pi}{12} &= 1\\ \cos^2 \tfrac{\pi}{12} - \sin^2 \tfrac{\pi}{12} &= \cos \tfrac{\pi}{6}.\end{aligned}$$

Adding these equations and dividing by 2 yields

$$\begin{aligned}\cos^2 \tfrac{\pi}{12} &= \tfrac{1}{2}\left(1+\cos\tfrac{\pi}{6}\right)\\ &= \tfrac{1}{2}\left(1+\tfrac{\sqrt{3}}{2}\right).\end{aligned}$$

Similarly, subtraction and division by 2 yields

$$\sin^2 \tfrac{\pi}{12} = \tfrac{1}{2}\left(1-\tfrac{\sqrt{3}}{2}\right).$$

13. Claim. (Addition formula for tan)

$$\tan(b+a) = \frac{\tan b + \tan a}{1 - \tan b \tan a}$$

Proof.

$$\begin{aligned}\tan(b+a) &= \frac{\sin(b+a)}{\cos(b+a)}\\ &= \frac{\sin b \cos a + \cos b \sin a}{\cos b \cos a - \sin b \sin a}\end{aligned}$$

The result follows by dividing both numerator and denominator by $\cos b \cos a$.

14. Corollary.

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t}.$$

Exercises

Philip Pennance[2] Version: – February 2021.

1. Halle el valor exacto de:

   (a) $\cos(\pi/12)$

   (b) $\cos\dfrac{\pi}{12}$

   (c) $\cos 120^\circ$

   (d) $\dfrac{2\tan(\pi/12)}{1-\tan^2(\pi/12)}$

   (e) $\sin\dfrac{\pi}{8}$

   (f) $\dfrac{2\tan(\pi/12)}{1-\tan^2(\pi/12)}$

   (g) $\sin 150^\circ$

2. Halle el valor exacto de:

   (a) $\sin 70^\circ \cos 20^\circ - \cos 70^\circ \sin 20^\circ$.

   (b) Encuentre el valor exacto de: $\cos\left(\frac{7\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right) - \sin\left(\frac{7\pi}{13}\right)\sin\left(\frac{6\pi}{13}\right)$.

   (c) $\cos\left(\frac{\pi}{4} - \cos^{-1}\left(\frac{3}{5}\right)\right)$

   (d) $\sin(\frac{\pi}{2} - \sqrt{2})\cos(\frac{\pi}{2} + \sqrt{2})$
   $+ \cos(\frac{\pi}{2} - \sqrt{2})\sin(\frac{\pi}{2} + \sqrt{2})$

3. Si $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ halle el valor exacto de $\cos\frac{4\pi}{5}$.

4. Halle el valor exacto de

$$\sum_{n=1}^{7} \cos\left(\frac{n\pi}{8}\right)$$

5. Si $\frac{\pi}{2} < \alpha < \pi$ y $|\sec\alpha| = \frac{17}{8}$, halle

   (a) $\sin 2\alpha$

   (b) $\tan 2\alpha$

   (c) $\sin\frac{\alpha}{2}$

   (d) $\tan\frac{\alpha}{2}$

6. Exprese sin el uso de funciones trigonométricas o funciones trigonométricas inversas

   (a) $\cos\left(2\cos^{-1} x\right)$. (b) $\tan\left(2\tan^{-1} x\right)$.

7. Si $\csc t = -\dfrac{13}{12}$ y $\cos t > 0$ halle:

   (a) $\cos t$ (b) $\tan t$ (c) $\sin 2t$

8. Halle $\sin(a-b)$ si $\tan a = 1/4$, $\cos b = 1/3$, y $0 < a, b < \pi/2$.

9. Halle $\cos(a+b)$ si $\cot a = 4/3$ y $\sec b = 13/12$ y $0 < a, b < \pi/2$.

10. Demuestre:

$$(\cos x + i\sin x)^2 = \cos 2x + i\sin 2x$$

11. Verifique las identidades siguientes:

    (a) $1 - \sec x \cos^3 x = \sin^2 x$

    (b) $1 - \sec x \cos^3 x = \sin^2 x$.

    (c) $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x} = 1$.

    (d) $\cot x \sin x - \cos^2 x \sec x = 0$

    (e) $\dfrac{\sin(x+y)}{\sin x \cos y} = 1 + \cot x \tan y$

    (f) $\tan\dfrac{t}{2} = \dfrac{\sin t}{1+\cos t}$.

    (g) $\tan x + \cot x = \sec x \csc x$.

    (h) $\frac{1}{2}\left[\sin(s-t) + \sin(s+t)\right]$
    $= \sin s \cos t$.

    (i) $\sqrt{1+\sin 2x} = |\sin x + \cos x|$

    (j) $(\sin x)(\tan x) = \sec x - \cos x$.

    (k) $(a\cos x + b\sin x)^2$
    $+ (b\cos x - a\sin x)^2 = a^2 + b^2$.

    (l) $\dfrac{1+\sin t}{1-\sin t} = (\sec t + \tan t)^2$

12. Halle constantes a y b tal que $\sin\left(x + \frac{7\pi}{2}\right) = a\sin x + b\cos x$

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13. Se conoce que $(\cos 1 + i \sin 1)^2 = \cos 2 + i \sin 2$. Utilice esta ecuación para demostrar que

$$(\cos 1 + i \sin 1)^3 = \cos 3 + i \sin 3.$$

14. (a) Demuestre para todo n natural:

$$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta).$$

(b) Halle una fórmula para $\sin 3x$ en términos de $\sin x$.

(c) Halle una fórmula para $\cos 3x$ en términos de $\cos x$.

15. Escriba $\cos 2\phi \cos 3\phi$ como una suma.