The Solution of Triangles.

Philip Pennance[1] – May 2021.

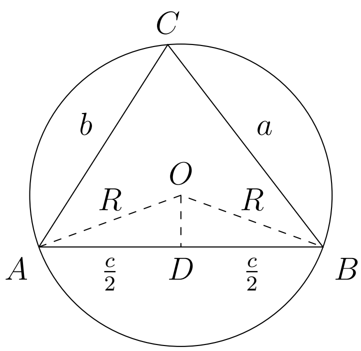
1. Claim. (Sine Rule.)

   In any triangle $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

   where R is the radius of the circumscribing circle.

   Proof.



   Let O be the center of the circumscribed circle. Then O can be either inside, outside, or on the triangle. If O is inside (see picture), then $\triangle\, AOB$ is isoceles. Let D be the foot of the perpendicular from O to AB, Then OD bisects both angle $\angle\, AOB$ and the side AB, and so $AD = \frac{c}{2}$. By a theorem of Euclid, the angle subtended by an arc at the center, of a circle is twice the angle subtended at the circumference. It follows that $\angle\, ACB = \frac{1}{2}\angle\, AOB$ and so $C = \angle\, ACB = \angle\, AOD$. Therefore,

$$\begin{aligned} \sin\ C &= \sin\ \angle\, AOD \\ &= \frac{AD}{OA} \\ &= \frac{\frac{c}{2}}{R} \\ &= \frac{c}{2R}. \end{aligned}$$

   Hence $2\,R = \dfrac{c}{\sin\ C}$. The same argument shows that $\dfrac{a}{\sin A} = 2R$ and $\dfrac{b}{\sin B} = 2R$.

   The cases when O is outside, or on the triangle can be proven in a similar manner.

2. Corollary.

   The area of $\triangle ABC$ is $\dfrac{abc}{4R}$.

   Proof.

   The area of $\triangle ABC$ is $\frac{1}{2}bc\sin A$. By the sin rule

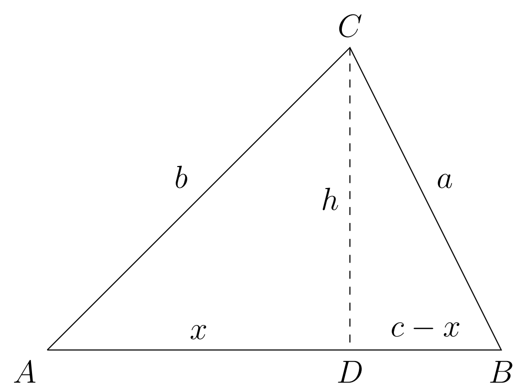
$$\sin A = \frac{a}{2R}$$

   and the result follows immediately.

3. The cosine rule.

   In any triangle $\triangle ABC$,

$$a^2 = b^2 + c^2 – 2bc\cos A$$



   Proof.

   Let h be the height of triangle as shown. Since $\triangle BCD$ is right triangled, it follows from the Pythagorean theorem that

$$\begin{aligned} h^2 &= a^2 - (c – x)^2 \\ &= a^2 - c^2 + 2cx + x^2. \end{aligned}$$

   Similarly, using $\triangle ADC$

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$$h^2 = b^2 - x^2.$$

Comparing the two expressions and noting that $\cos A = \frac{x}{b}$ yields

$$\begin{aligned} a^2 &= c^2 – 2cx + b^2 \\ &= c^2 – 2cb \cos A + b^2. \end{aligned}$$

and the result follows.

4. Corollary

$$\begin{aligned} a^2 &= b^2 + c^2 – 2bc \cos A \\ b^2 &= a^2 + c^2 – 2ac \cos B \\ c^2 &= a^2 + b^2 – 2ab \cos C. \end{aligned}$$

Exercises

Prof. Philip Pennance[2] -Version: January 16, 2017

1. Un triangulo tiene lados $a = 5$, $c = 13$ y $b = 12$. Si b es el lado opuesto al ángulo β.
   (a) Halle $\sin \beta$ y $\cos \beta$.
   (b) Verifique que $(\sin \beta)^2 + (\cos \beta)^2 = 1$.

2. En $\triangle ABC$, $\angle A$ tiene medida $90°$, $b = 6$ y $\tan \angle B = \frac{3}{4}$. Halle:
   (a) a
   (b) c
   (c) $\cos \angle B$
   (d) $\tan \angle C$

3. En $\triangle ABC$, $\angle A$ tiene medida $90°$, $b = 7$ y $\tan \angle B = \frac{1}{4}$. Halle
   (a) a
   (b) c
   (c) $\sec \angle B$
   (d) $\tan \angle B$
   (e) $\cot \angle C$

4. $\triangle ABC$ tiene lados de medidas $6, 10$ y 12. Halle el coseno del ángulo más grande.

5. Un triangulo T tiene lados $a = 5$, $c = 13$ y $b = 12$.
   (a) Demuestre que T es rectángulo.
   (b) Si β es el ángulo con lado opuesto b halle $\sin \beta$ y $\cos \beta$.
   (c) Verifique que
   $$(\sin \beta)^2 + (\cos \beta)^2 = 1$$

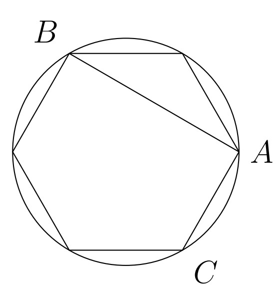
6. Un triangulo tiene lados $2, 3, 4$. Halle el coseno del ángulo opuesto el lado de medida 3.

7. Un circulo pasa por los vertices del $\triangle ABC$. Si $\angle A = \frac{\pi}{4}$ y $a = 12$, halle el diametro del circulo.

8. Un circulo de radio $R = 8$ pasa por los vertices del $\triangle ABC$. Si el triángulo tiene area 10, halle el producto abc.

9. Un circulo pasa por los vertices del $\triangle ABC$. Halle la medida del lado a si el circulo tiene radio $R = 5$ y $\angle A = 2\pi/3$ (radians).

10. Un hexágono regular está inscrito en un círculo de radio 1.



Halle las medidas de los segmentos AB, y AC. Demuestre que $\triangle ABC$ es un triángulo rectángulo.

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