The Square of Opposition

Prof. Pennance[1] Version: – November 11, 2024.

1. Definition of a set by specification.

   Let U be a set and $p(x)$ a predicate valid for the set U. Then $\{x \in U : p(x)\}$ defines a set, the elements of which are precisely those elements of the set U for which the predicate is true. This method of defining sets is called *specification*. The set U is called the *ground set* or *universe*. See (Notes on Set Theory).

2. Example.

   Let $s(x)$ be the predicate "x is a saint" and $p(x)$ the predicate "x is a politician." The set S of saints and the set P of politicians are defined respectively by

   (a) $S = \{x \in U : s(x)\}$ (b) $P = \{x \in U : p(x)\}$

   The complements S' and P' of these two sets can be written:

   (a) $S' = \{x \in U : \neg s(x)\}$ (b) $P' = \{x \in U : \neg p(x)\}$

3. Definition.

   Let $s(x)$ and $p(x)$ be any predicates. The classical categorical propositions determined by s and p, namely:

   (a) Universal Affirmative:
   $\forall x \in S, \quad p(x)$

   (b) Particular Affirmative:
   $\exists x \in S, \quad p(x)$

   (c) Universal Negative:
   $\forall x \in S, \quad \neg p(x)$

   (d) Particular Negative:
   $\exists x \in S, \quad \neg p(x)$

   will be referred to, in what follows, as *sp propositions.* The above four forms of *sp* proposition are called *moods* and are traditionally denoted, respectively, by the letters A, I, E, O. These letters are the initial 2 vowels of the latin words "[a]f[i]rmo" –to affirm and "n[e]g[o]" – to deny.

4. Example.

   Let $s(x)$ be the predicate "x is a saint" and $p(x)$ the predicate "x is a politician." The four moods are expressed linguistically by

   (A) Every saint is a politician

   (I) Some saint is a politicion

   (E) No saint is a politician

   (O) Some saint is not a polition

   or, more briefly:

[1] https://pennance.us. Reference: https://www.wikidoc.org/index.php/Square_of_opposition

(A) Every s is p	(E) No s is p
(I) Some s is p	(O) Some s is not p

5. The four moods have the following interpretations in terms of sets:

(A)	Every saint is a politician.	$S \subseteq P$.
(I)	Some saint is a politicion.	$S \cap P \neq \emptyset$.
(E)	No saint is a politician.	$S \subseteq P'$.
(O)	Some saint is not a politicion.	$S \cap P' \neq \emptyset$.

   Remark. In classical logic, it is tradionally assumed that the set S is non-empty. In what follows we make this assumption. This will mean that some statements which are true in classical propositional logic may not be in modern logic. Discussions of this point can be found in the references.

6. Logical Notation.

   In mathematical logic, the moods (A) and (E) can also be written in terms of implication ($\rightarrow$) and (I) and (O) using conjunction $\wedge$.

(A)	Every man is mortal.	$\forall x$ (x is a man) $\rightarrow$ (x is mortal).
(E)	No man is mortal.	$\forall x$ (x is a man) $\rightarrow \neg$ (x is mortal).
(I)	Some man is mortal.	$\exists x$ (x is a man) $\wedge$ (x is mortal).
(O)	Some man is not mortal.	$\exists x$ (x is a man) $\wedge \neg$ (x is mortal).

7. Definition.

   The *quality* of a proposition refers to whether the proposition is affirmative or negative.

8. Examples.

   The propositions "All dogs are mammals." "Some birds can fly" have Affirmative Quality, whereas "No reptiles have feathers." "Some metals do not conduct electricity" have Negative Quality

9. Definition.

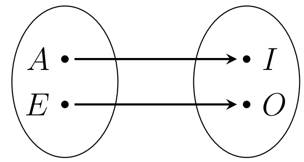
   The proposition obtained by replacing the universal quantifier by an existential quantifier in a proposition of type A or E is called the *subaltern* or *corresponding particular proposition.*

10. Example.

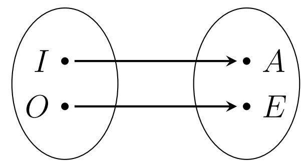
    Universal affirmative: All birds can fly. (A proposition)
    Subaltern: Some birds can fly. (I proposition)

11. Definition.

    Subalternation is the function with domain {A,E} and codomain {I,O} which maps each universal proposition to its subaltern.



The inverse function (drawn below) is called *superalternation*.



and so, the universal affirmation A is the superaltern of the corresponding particular I. The the universal negation E is the superaltern of the corresponding particular negation O.

12. Properties of the subalternation and superalternation.

    Let $s(x)$ and $p(x)$ be any predicates valid on some set U and let A,E, I and O be the corresponding *sp* propositions:

(A)	Every s is p.	$\forall x$	$(s(x) \rightarrow p(x))$.
(E)	No s is p.	$\forall x$	$\neg(s(x) \rightarrow p(x))$.
(I)	Some s is p.	$\exists x$	$(s(x) \wedge \ p(x))$.
(O)	Some s is not p.	$\exists x$	$(s(x) \wedge \neg p(x))$.

    If the set $S = \{x \in U : s(x)\}$ is non empty, then the following implications are true:

    (a) $A \implies I$ (b) $E \implies O$ (c) $\neg I \implies \neg A$ (d) $\neg O \implies \neg E$

    Proof of (a).

    We must show that I is true, whenever A is true. Therefore, suppose that A is true. Then $\forall x \quad (s(x) \rightarrow p(x))$ is true. Since S is non empty, There exists an element $x_0 \in U$ such that $s(x_0)$ is true and for this element $(s(x_0) \rightarrow p(x_0)$ is true. It follows by modus ponens that $p(x_0)$ is also true which means that I is true. Hence, the implication $A \implies I$ is true.

    The proof of (b) is similar. The properties (c) and (d) of superalternation follow from (a) and (b) by the law of contraposition.

13. Definition.

    Let F and G be propositions. We say that G is a *contradictory* of F if G is equivalent to the negation of F. I.e., $G \equiv \neg F$.

14. Lemma.

    The following are equivalent.

(a) G is a contradictory of F

(b) F is a contradictory of G.

Proof.

Suppose that G is a contradictory of F. Then, using double negation,

$$\begin{aligned} G \equiv \neg F &\iff \neg G \equiv \neg\neg F \\ &\iff \neg G \equiv F \\ &\iff F \equiv \neg G \end{aligned}$$

It follows that F is a contradictory of G if and only if G is a contradictory of F.

15. Definition.

Two propositions are *contrary* if they cannot be both true. If (F, G) is a pair of contrary propositions, we say that G *is a contrary of* F or equivalently F *is a contrary of* G

Remark. The definition does not preclude the possibility that both may be false!)

16. Claim. (An alternative characterization of the notion of contrary)

Let F and G be propositions. The following are equivalent.

(a) F and G are contrary.

(b) $F \implies \neg G$ (or equivalently $G \implies \neg F$).

Proof.

Suppose that $F \implies \neg G$ is true. Then F and G cannot both be true, for this would contradict the truth of the implication $F \implies \neg G$. Hence, by definition, the two propositions are contrary.

Conversely, let F and G be contrary. We must show that $F \implies \neg G$. To this end, suppose that F is true. Since F and G cannot both be true, it can only be that G is false. It follows that $F \implies \neg G$.

17. Example.

Let A be the proposition "Every man is mortal," and E the proposition "No man is mortal." Then A and E are contraries (provided that the set of men is non empty). They cannot both be true.

18. Definition.

Two propositions are *subcontrary* if they cannot both be false. If (F, G) is a pair of contrary propositions, we also say that G *is a subcontrary of*bF or equivalently F *is a contrary of* G (The definition does not preclude the possibility that both may be true!)

19. Claim. (An alternative characterization of the notion subcontrary)

Let F and G be propositions. The following are equivalent.

(a) F and G are subcontrary.

(b) $\neg F \implies G$ (or equivalently $\neg G \implies F$.

Proof.

Suppose that $\neg F \implies G$ is true. Then F and G cannot both be false, for this would contradict the truth of the implication $\neg F \implies G$. Hence, by definition, the two propositions are subcontrary.

Conversely, let F and G be subcontrary. We must show that $\neg F \implies G$. To this end, suppose that $\neg F$ is true or equivalently, that F is false. Since F and G are subcontrary, they cannot both be false. It can only be that G is true. It follows that $\neg F \implies G$.

20. Example.

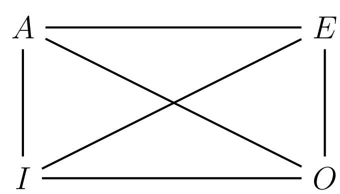
If the set of students is non empty, the propositions "some students are polite", and "some students are not polite" are sub-contrary since they clearly cannot both be false.

21. Definition.

Let $s(x)$ and $p(x)$ be any predicates and let A,E, I and O be the following *sp*–propositions (i.e., with subject s and predicate p):

(A)	Every s is p.	$\forall x$	$(s(x) \to sp(x))$.
(E)	No s is p.	$\forall x$	$\neg(s(x) \to p(x))$.
(I)	Some s is p.	$\exists x$	$(s(x) \wedge\ p(x))$.
(O)	Some s is not p.	$\exists x$	$(s(x) \wedge \neg p(x))$.

The *square of opposition* is the *graph* with vertex set $\{A, E, I, O\}$ and edge set as shown in the following drawing:



22. Properties of the Square of Opposition.

If $S = \{x : s(x)\}$ is a non empty set (I.e., $\exists x\, s(x)$.)

(a) (A, O) and (I, E) are pairs of contradictories.

(b) (A, E) is a pair of contraries.

(c) (I, O) is a pair of subcontraries.

Proof of (a).

Suppose that A is false. This is equivalent to the statement

"it is not the case that for all x the implication $(s(x) \implies p(x))$ is true" which is equivalent to

"for some x the implication $s(x) \implies p(x)$ is false"

By the definition of implication, this is equivalent to

"for some x_0, $s(x_0)$ is true and $p(x_0)$ is false"

which in turn is equivalent to O.

It follow that A is the negation of O and so A and O are contradictories. Here is the same argument in symbols:

$$\begin{aligned} \neg A &\equiv \neg\forall x(s(x) \implies p(x)) \\ &\equiv \exists x, (\neg(s(x) \implies p(x)) \\ &\equiv \exists x_0, (s(x_0) \wedge \neg p(x_0)) \\ &\equiv O. \end{aligned}$$

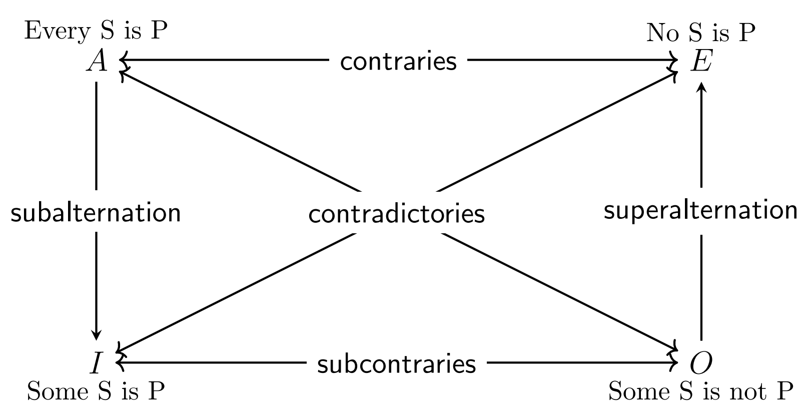
Proof of (b)

To show that E is the contrary of A, it suffices to show that if A is true, then E is false. Suppose, therefore, that A is true. Since I is the subaltern of A and S is non empty, we know that $A \implies I$. Since A is true and $A \implies I$ is true, it follows by modus ponens that I is true. But I and E are contradictories and so E must be false, which was to be proven.

Proof of (c)

To show that O and I are subcontraries, by item (19) above, it suffices to show that $\neg O \implies I$. Suppose therefore that $\neg O$ is false. Since O is the subaltern of E, we have by the subaltern property (12) that $\neg O \implies \neg E$ and it follows that E is false. But E is a contradictory of I and therefore I must be true. This, O and I are subcontraries.

The statements are summarized in the following diagram:



Remark.

For simplicity, the superalternation from $I \to A$ and the subalternation from $E \to O$ are ommitted from the diagram.

23. Summary.

    If set S is non-empty, i.e., if $\exists x, \quad s(x)$, then the contradictory, contrary, subcontrary, subalternation, and superalternation relations have the following properties:

    (a) Contradictary Relations

        i. $A \iff \neg O.$ ii. $E \iff \neg I.$

    (b) Contrary Relations

        i. $A \implies \neg E.$ ii. $E \implies \neg A.$

    (c) Subcontrary Relations

        i. $\neg I \implies O$ ii. $\neg O \implies I$

    (d) Subalternation Relations

        i. $A \implies I$ ii. $E \implies O$

    (e) Superalternation Relations

        i. $\neg I \implies \neg A$ ii. $\neg O \implies \neg E$

Online References

1. Wikidoc.org, Article on the Square of pposition.
2. Article in the Stanford Encyclopedia of Philosophy.
3. International Congress on the Square of Opposition.