The Synthetic Substitution/Division Algorithm

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Claim.

Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial in $\mathbb{C}[x]$ and let

$$q(x) = q_0x^{n-1} + q_1x^{n-2} + \cdots + q_{n-1}$$

where the constants $q_0, q_1, \ldots, q_n$ are determined by the following algorithm:

Algorithm: `Synthetic Division`
Input: a_0, a_1, a_2, ..., a_n, α
Initialization: $q_0 := a_0$
Iteration : $q_i := \alpha q_{i-1} + a_i$, $1 \le i \le n$
Output: $q_0, q_1, \ldots, q_n$

Then,

$$p(x) = (x-\alpha)q(x) + q_n. \tag{1}$$

I.e., the synthetic substitution algorithm determines both the quotient $q(x)$ and remainder r when $p(x)$ is divided by $x - \alpha$.

Proof. (Special case: n=3.)

$$\begin{aligned} p(\alpha) &= a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 \\ &= \alpha\left[a_0\alpha^2 + a_1\alpha + a_2\right] + a_3 \\ &= \alpha\left[\alpha(a_0\alpha + a_1) + a_2\right] + a_3 \\ &= \alpha\left[\alpha q_1 + a_2\right] + a_3 \\ &= \alpha q_2 + a_3 \\ &= q_3. \end{aligned}$$

and so $r = q_3$. Also,

$$\begin{aligned} p(x) - p(\alpha) &= a_0(x^3 - \alpha^3) + a_1(x^2 - \alpha^2) + a_2(x - \alpha) \\ &= (x-\alpha)\left[a_0(x^2 + \alpha x + \alpha^2) + a_1(x + \alpha) + a_2\right] \\ &= (x-\alpha)\left(a_0x^2 + (a_0\alpha + a_1)x + a_0\alpha^2 + a_1\alpha + a_2\right). \end{aligned}$$

It follows that

$$p(x) = (x-\alpha)\left(a_0x^2 + (\alpha a_0 + a_1)x + a_0\alpha^2 + a_1\alpha + a_2\right) + r.$$

Comparing coefficients with (1) yields $q_0 = a_0$, $\quad q_1 = \alpha q_0 + a_1$ and,

$$\begin{aligned} q_2 &= a_0\alpha^2 + a_1\alpha + a_2 \\ &= \alpha(\alpha q_0 + a_1) + a_2 \\ &= \alpha q_1 + a_2. \end{aligned}$$

in accordance with the algorithm.

[1] https://pennance.us

Rational Root Theorem

Philip Pennance[2] Version: – March, 2021.

1. Let $r = \frac{\alpha}{\beta}$ be a rational fraction in lowest terms and p a polynomial with integer coefficients. We seek a necessary condition for r to be a root of p.

2. Example.
   Let
   $$p(x) = 6x + 4.$$
   Then $p(\frac{-2}{3}) = 0$. Notice that the numerator -2 of the root is a factor of the constant term of p and that the denominator 3 is a factor of the leading term. This is not an accident.

3. Example.

   Let
   $$p(x) = 2x^2 + x - 6.$$
   Then $p\left(\frac{3}{2}\right) = 0$. Notice that the numerator 3 of the root is a factor of the constant term of p and that the denominator 2 is a factor of the leading term. Neither is this is not an accident.

4. Rational root theorem.
   Let
   $$a_0x^n + a_1x^{n-1} + \cdots + a_n$$
   be a polynomial with integer coefficients. If $\frac{\alpha}{\beta} \in \mathbb{Q}$ is a root and $\gcd(\alpha, \beta) = 1$, then the numerator α is a factor of the constant term a_n and the denominator β divides the leading coefficient a_0.

   Proof. $n = 3$ (The general proof is exactly the same). Let
   $$p(x) = a_0x^3 + a_1x^2 + a_2x + a_3.$$
   where $a_0, a_1, a_2, a_3 \in \mathbb{Z}$. Let $\frac{\alpha}{\beta} \in \mathbb{Q}$ be a fraction in lowest terms such that $p\left(\frac{\alpha}{\beta}\right) = 0$. Then
   $$a_0\left(\frac{\alpha}{\beta}\right)^3 + a_1\left(\frac{\alpha}{\beta}\right)^2 + a_2\left(\frac{\alpha}{\beta}\right) + a_3 = 0.$$
   Multiplication by β^3 gives:
   $$a_0\alpha^3 + a_1\alpha^2\beta + a_2\alpha\beta^2 + a_3\beta^3 = 0.$$
   Hence
   $$a_0\alpha^3 + a_1\alpha^2\beta + a_2\alpha\beta^2 = -a_3\beta^3.$$
   It follows that α is a factor of $a_3\beta^3$. Since $gcf(\alpha, \beta) = 1$ it must be that α is a factor of the constant term a_3. A similar argument shows that β is a factor of a_0.

5. Example.

   Show that $x = \sqrt[3]{7}$ is not rational.

   Solution.

   Notice that x is a root of the polynomial $p(x) = x^3 - 7$. By the rational root theorem, the only candidates for rational roots are $\pm 7, \pm 1$. But none of these are roots and so $\sqrt[3]{7}$ cannot be rational.

[2] https//pennance.us

Exercises

1. Sea

$$f(x) = x^4 + 14x^3 + 12x^2 + 87x + 5$$

Utilice división sintética para hallar $f(-13)$.

2. Identify the mistake (if any) in the synthetic division of $x^4 + 3x^2 - 4x - 2$ by the factor $x - 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -2 \\ & & 2 & 10 & 12 \\ \hline & 1 & 5 & 6 & 10 \end{array}$$

3. Sean $p(x) = x^3 - 7x^2 + 5x + 3$ y $D(x) = x - 1$.

   (a) Utilice sintética para encontrar el cociente y el residuo al dividir $P(x)$ por $D(x)$

   (b) Encuentre números A, B , C, y D tal que

$$\frac{P(x)}{D(x)} = Ax^2 + Bx + C + \frac{D}{x-1}$$

4. Escriba cada uno de los polinomios siguientes como un producto de factores irreducibles sobre los complejos, los reales y los racionales.

   (a) $p(x) = x^3 - 5x^2 + 11x - 15$.
   Ans. $(x-3)(x^2 - 2x + 5)$

   (b) $p(x) = x^4 - 4x^3 + x^2 + 12x - 12$.

   (c) $p(x) = 2x^4 + 5x^3 + 3x^2 + 15x - 9$.

   (d) $p(x) = 2x^4 - 7x^3 + 5x^2 + 9x - 5$
   Ans. $(x+1)(2x-1)(x^2 - 4x + 5)$.

5. Demuestre que no existen enteros a, b tales que $5^{2/7} = a/b$

6. Encuentre todas las soluciones de

$$x^4 + x^3 - 7x^2 - x + 6 = 0$$

Ans. { 2,1,-1,-3 }

7. Halle el cociente y residuo al dividir $x^3 - 8$ por $x - 3$.

8. Encuentre el cociente $Q(x)$ y el residuo $R(x)$ al dividir al polinomio $P(x) = 2x^4 - 3x^3 + 10x^2 - x + 2$ por $D(x) = x^2 - 5x + 1$.

9. Halle b si una de las soluciones de la ecuación $x^2 + bx + 1 = 0$ es dos veces la otra. Ans. $b = \pm\frac{3}{2\sqrt{2}}$.

10. Si r_1 y r_2 son soluciones de la ecuación $x^2 + bx + c = 0$ demuestre que $r_1 r_2 = c$ y $r_1 + r_2 = -b$.

11. Si $x + 2$ es un factor de

$$p(x) = x^2 + ax + b$$

halle $p(1)$ si $p(5) = 5$.

12. Una solución de la ecuación

$$x^3 - 8x^2 + 16x - 3 = 0$$

es 3. Halle la suma de las otras soluciones. Ans. 5

13. Enuncie el teorema fundamental del Algebra.

14. Resuelva $x^2 - 25 = 2x - 75$ sobre el sistema de números complejos.

15. Halle los valores de a y b si $x - 1$ es un factor de $p(x) = x^3 + x^2 + ax + b$ y de $q(x) = x^3 - x^2 - ax + b$.

16. Halle k si $x - 2$ es un factor de

$$x^3 + (3-k)x^2 + kx - 6$$

17. Halle un polinomio de grado mínimo y con coeficientes reales si $1/3$ y $1 + i$ son ceros del polinomio.

18. Halle un polinomio con coeficientes enteros y grado mímo si $-1, i$ son ceros.

19. Halle un polinomio p con coeficientes reales y grado mínimo si $1-i$ y 3 son soluciones de $p(x)=0$.

20. Halle un polinomio de grado 3 y con coeficientes enteros si $-1/2$, y $3+4i$ son ceros.

21. Sea $p(x)=x^3-4x^2+3x-12$.

    (a) Resuelva la ecuación $p(x)=0$ si $i\sqrt{3}$ es una solución.

    (b) Escriba p como un producto de factores irreducibles sobre C.

22. Factorice completamente sobre los números complejos, el polinomio

$$p(x)=x^4+3x^3-63x^2+249x-286$$

si $\alpha=3+2i$ es una raíz de p.
Ans. $(x^2-6x+13)(x-2)(x+11)$.

23. Si $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ es una raíz de

$$x^4-6x^2-7x-6$$

encuentre todos sus ceros.

24. Encuentre el residuo al dividir

$$x^{1000}+x^{100}-3x^{12}+1$$

entre $x-i$.