The Plane

Philip Pennance[1] – February 2019.

1. Definition.

   The *real plane* (denoted $\mathbb{R}^2$) is the set of all ordered pairs (x, y) of real numbers.

2. Example.

   Let $P = (3, 2\sqrt{2})$. Then P is an element of $\mathbb{R}^2$ since both 3 and $2\sqrt{2}$ are real numbers.

3. Definition.

   A set of the form $\{(x, y) \in \mathbb{R}^2 : x = a\}$ where a is constant is called a *vertical line*.

4. A set of the form $\{(x, y) \in \mathbb{R}^2 : y = b\}$ is called a *horizontal line*.

5. Definition.

   Let $P = (x, y)$ be a point in the plane. Then x is called the *abscissa* of P and y the *ordinate*.

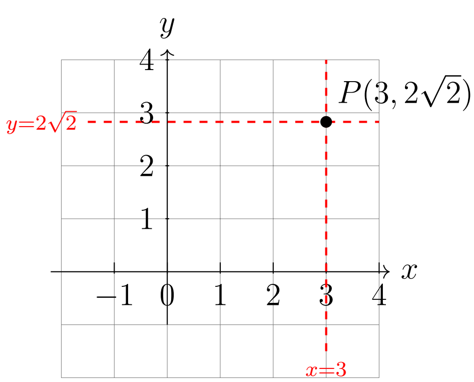
6. Geometric Representation.

   A point $P = (a, b)$ can be drawn in the usual manner using two perpendicular coordinate axes as show. It is located at the intersection of the lines $x = a$ and $y = b$.

7. Example.

   Draw the point $P = (3, 2\sqrt{2})$.

   Solution.

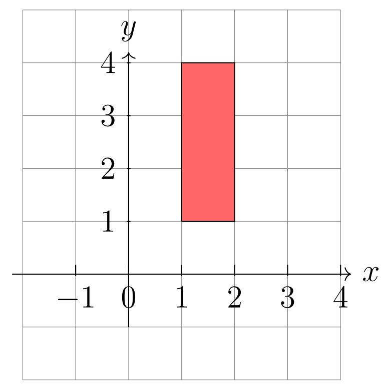


8. Example.

   Draw the set $[1, 2] \times [1, 4]$.

   Solution.

   Notice that $(x, y) \in [1, 2] \times [1, 4]$ if and only if $1 \le x \le 2$ and $1 \le y \le 4$. Hence the graph is a rectangle.



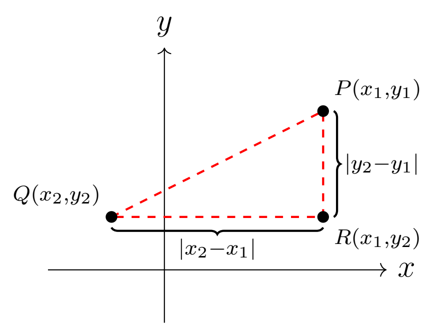
9. Claim.

   Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points in the plane. The distance $d(P, Q)$ between P and Q

   $$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

   Proof.

   Let R be the point (x_1, y_2). It is easily verified that $\triangle PQR$ is right angled.



[1] https://pennance.us

By the Pythagorian theorem,

$$\begin{aligned} d(P, QP &= \sqrt{QR^2 + RP^2} \\ &= \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \end{aligned}$$

10. The Midpoint Formula.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in the plane. The midpoint M of the segment PQ is given by

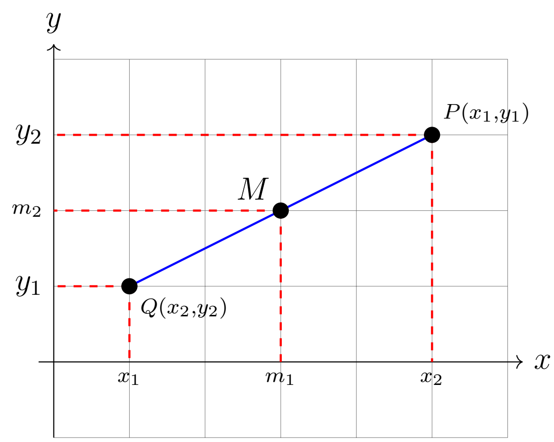
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Proof.

Let $M = (m_1, m_2)$. By similar triangles (exercise) the projection m_1 of M onto the horizontal axis must be the midpoint of the projections of P and Q. Hence by the 1-dimensional midpoint formula

$$m_1 = \left(\frac{x_1 + x_2}{2} \right)$$

Similarly, projection onto the vertical axis yields the ordinate m_2 of the midpoint.



Exercises

1. Halle $x, y \in \mathbb{R}$ si
$$(x + y, x - y) = (1, -2).$$
2. Halle todos los puntos $(x, y) \in \mathbb{R}^2$ tal que
$$(x^2 + y^2, x^2 - y^2) = (5, 3).$$
3. Dibuje el conjunto de puntos $(x, y) \in \mathbb{R}^2$ que cumplen
$$(x + y, x - y) = (x + y, -2).$$
4. Halle el punto $(x, y, z) \in \mathbb{R}^3$ tal que
$$(x^2 + y^2 + z, z) = (4, 4).$$
Justifique su respuesta.
5. Halle una ecuación del bisector perpendicular de $(-1, \pi)$ y $(\pi, -1)$.
6. Dibuje los conjuntos siguientes:
   (a) $\{x \in \mathbb{R} : x \leq 1\}$
   (b) $\{(x, y) \in \mathbb{R}^2 : x \leq 1\}$
7. Dibuje los conjuntos siguientes:
   (a) $\{y \in \mathbb{R} : y \geq 2\}$
   (b) $\{(x, y) \in \mathbb{R}^2 : y \geq 2\}$
8. Dibuje los conjuntos siguientes:
   (a) $\{(x, y) \in \mathbb{R}^2 : |x| \leq 2 \wedge |y| \leq 3\}$
   (b) $\{(x, y) \in \mathbb{R}^2 : |x| \leq 2 \wedge |y| \geq 3\}$
9. Dibuje los conjuntos siguientes:
   (a) $\{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\}$
   (b) $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$
10. Un triangulo equilatero tiene vertices $(0, 0)$, $(a, 0)$, (m, n). Si m, n son enteros, demuestre que a no es un entero.
11. Sea $P = (2, 3)$ and $Q = (-1, -2)$. Halle el punto medio del segmento PQ.

12. Halle los puntos en el eje x que están equidistante de los puntos con coordenadas $(2, 2)$ y $(6, 6)$.

13. Halle el punto equidistante de $(0, 3), (3, 4)$ y $(2, 1)$.

14. Sea $P = (3, 5)$ y $Q = (5, 3)$. Halle una ecuación de la recta que pasa por el punto medio de P y Q y es perpendicular al segmento PQ.

15. Halle la reflección del punto (a, b):

    (a) En el eje de x.

    (b) En el eje de y

    (c) En el origen,

    (d) En la recta $y = x$

    (e) En la recta $x = 2$

16. Halle el centroide del triángulo con vertices $(0, 3), (3, 4)$ y $(2, 1)$.