

# The Limit of a Sequence (Brief Summary)

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## 1. Definition.

A real number  $L$  is a *limit* of a sequence of real numbers if every open interval containing  $L$  contains all but a finite number of terms of the sequence.

## 2. Claim.

A sequence can have at most one limit.

Proof.

Suppose  $L$  and  $L'$  be limits of a sequence. Then there exists an open interval  $I$  containing  $L$  and an open interval  $I'$  containing  $L'$  and both of these intervals contain all but a finite number of terms of the sequence. If  $L \neq L'$ ,  $I$  and  $I'$  can be chosen to be disjoint which (exercise) leads to a contradiction. Thus, it must be that  $L = L'$ .

## 3. Notation.

If  $L$  is the limit of a sequence  $(a_n)$  we write

$$\lim_{n \rightarrow \infty} a_n = L$$

and we say that the sequence *converges* to  $L$  or that the sequence is *convergent*. A sequence with no limit is called *divergent*.

## 4. Remark.

The definition of limit gives precise meaning to the rather vague phrase “ $a_n$  approaches  $L$  as  $n$  approaches  $\infty$ ”. This statement does not serve as the definition of limit because neither the nature of the “approach” nor the concept of  $\infty$  are explained.

## 5. Lemma.

Let  $I$  is any open interval containing  $L$ , then within  $I$  can be found an open interval of the form  $|x - L| < \epsilon$

## 6. Theorem.

Let  $(a_n)$  be a sequence. The following are equivalent,

(a)  $\lim_{n \rightarrow \infty} a_n = L$

(b) For every  $\epsilon > 0$ , all but a finite number of terms of the sequence are contained in the interval with center  $L$  and radius  $\epsilon$ .

(c)  $\forall \epsilon > 0, \exists N$  such that  $\forall n \in \mathbb{N}$

$$n > N \implies |a_n - L| < \epsilon.$$

## 7. Example.

Let  $(a_n)$  be the sequence given by

$$a_n = \frac{n}{n+1}.$$

Show that  $\lim_{n \rightarrow \infty} a_n = 1$ .

Solution.

Let  $\epsilon > 0$  be arbitrary. We must find a natural number  $N$  such that<sup>1</sup>

$$d(a_n, 1) < \epsilon \tag{1}$$

whenever  $n > N$ . Now

$$d(a_n, 1) = \left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1}.$$

Moreover, since  $\mathbb{N}$  is unbounded above there exists a natural number  $N$  such that  $N\epsilon > 1$ . It follows that:

$$\begin{aligned} n > N &\rightarrow n > \frac{1}{\epsilon} \\ &\rightarrow \epsilon n > 1 \\ &\rightarrow \frac{1}{n} < \epsilon \\ &\rightarrow \frac{1}{n+1} < \epsilon \\ &\rightarrow d(a_n, 1) < \epsilon \end{aligned}$$

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<sup>1</sup>Notice that the value of  $N$  will depend on the value of  $\epsilon$ . The smaller the value of  $\epsilon$  the larger  $N$  must be in order that the open interval determined by equation (1) contains all terms after the  $N$ 'th. Figures 3 and 4 suggest that that if  $\epsilon = 0.2$  then  $N = 5$  works and if  $\epsilon = 0.1$  then  $N = 9$  suffices.

Since  $N$  has the required property, we conclude that

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

8. Example.

Let  $(a_n)$  be the constant sequence given by  $a_n = 5, n \in \mathbb{N}$ . Show that  $\lim_{n \rightarrow \infty} a_n = 5$ .

Solution.

Let  $\epsilon > 0$ . It is easy to find  $N$  such that  $|a_n - 5| < \epsilon$  whenever  $n > N$ . Let  $N = 24$  (any other natural number will do). Since  $|a_n - 5| = 0$ , for all  $n \in \mathbb{N}$  it is clearly true that  $|a_n - 5| < \epsilon$  whenever  $n > 24$ . It follows that  $\lim_{n \rightarrow \infty} a_n = 5$ .

9. More generally, if  $a_n = c$  for all  $n \in \mathbb{N}$  then  $\lim_{n \rightarrow \infty} a_n = c$ .

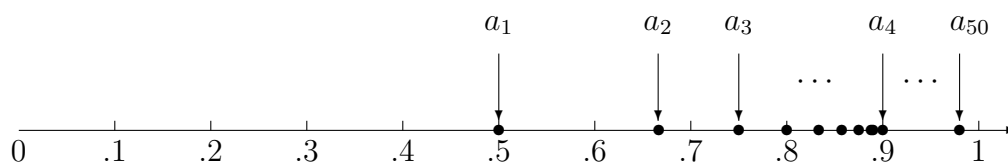


Figure 1. Initial Terms of the Sequence  $a_n = \frac{n}{n+1}$

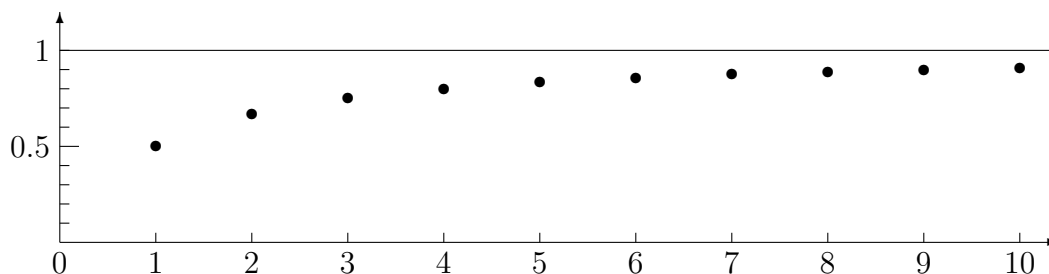


Figure 2. Graph of the sequence  $a_n = \frac{n}{n+1}$

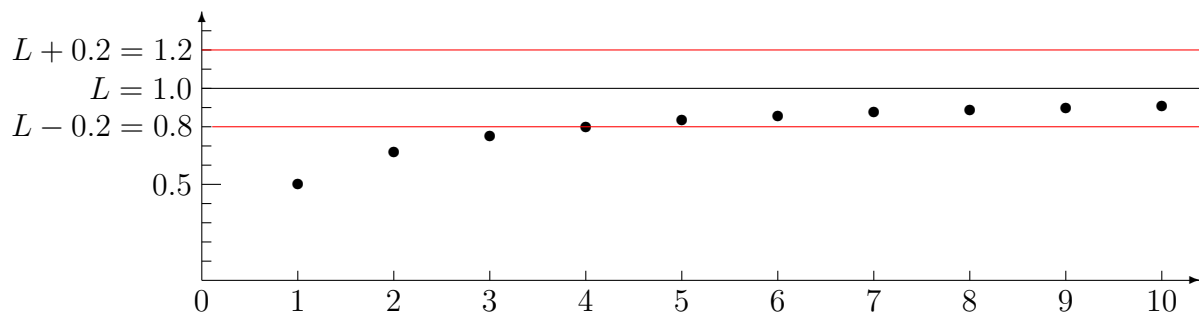


Figure 3.  $n > 5 \rightarrow |a_n - 1| < 0.2$ .

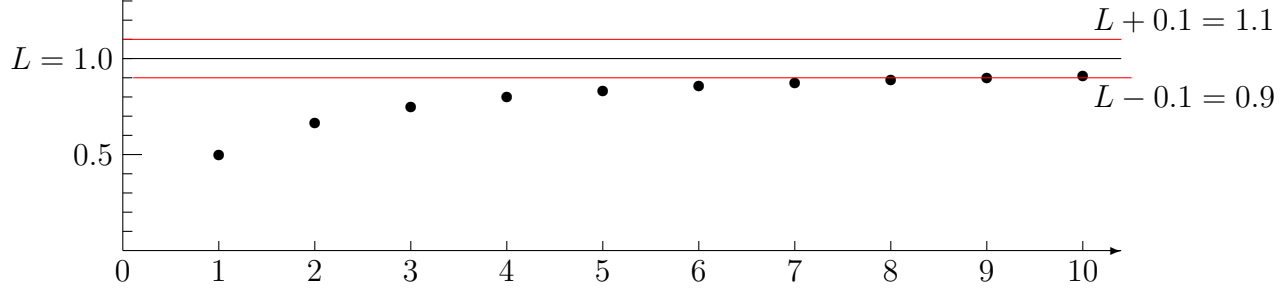


Figure 4.  $n > 9 \rightarrow |a_n - 1| < 0.1$ .

10. Example.

Let  $(a_n)$  be the sequence given by

$$a_n = \frac{1}{n-1}, \quad n > 1.$$

Show using the definition of the limit of a sequence that

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Solution.

Let  $\epsilon > 0$ . Notice (exercise) that  $\frac{1}{n-1} < \frac{2}{n}$  for all  $n > 2$ . Since  $\mathbb{N}$  is unbounded, there exists a natural number  $N > 1$  such that  $N\epsilon > 2$ . It follows that:

$$\begin{aligned} n > N &\rightarrow n\epsilon > 2 \\ &\rightarrow \frac{2}{n} < \epsilon \end{aligned}$$

Since  $\frac{1}{n-1} < \frac{2}{n}$ , it follows that

$$\begin{aligned} n > N &\rightarrow \frac{1}{n-1} < \epsilon \\ &\rightarrow \left| \frac{1}{n-1} - 0 \right| < \epsilon \\ &\rightarrow d(a_n, 0) < \epsilon \end{aligned}$$

which proves that

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} = 0$$

11. The next theorem reduces the calculation of complex limits to simpler ones.

12. Theorem.

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$  then:

$$(a) \quad \lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm M$$

$$(b) \quad \lim_{n \rightarrow \infty} (a_n \cdot b_n) = L \cdot M$$

(c) If  $M \neq 0$  and  $b_n \neq 0$  for all  $n$  then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}.$$

13. Example.

Find the limit of the sequence  $(a_n)$  given by

$$a_n = 3/n$$

Solution.

Notice  $a_n = x_n \cdot y_n$  where  $x_n = 1/n$  and  $y_n = 3$ ,  $n \in \mathbb{N}$ . Now, it is easy to show that  $\lim_{n \rightarrow \infty} x_n = 0$  and  $\lim_{n \rightarrow \infty} y_n = 3$ . Using part (b) of the above theorem yields

$$\lim_{n \rightarrow \infty} a_n = 0 \cdot 3 = 0$$

14. Example.

Find the limit of the sequence  $(a_n)$  given by

$$a_n = \frac{1}{n^2}.$$

Solution.

Let  $x_n = 1/n$ . Then

$$a_n = x_n \cdot x_n$$

Since  $\lim_{n \rightarrow \infty} x_n = 0$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} x_n \\ &= 0 \cdot 0 \\ &= 0. \end{aligned}$$

15. Using mathematical induction (exercise) it is now easy to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$$

for any natural number  $p > 0$ .

16. Example.

Find the limit of the sequence

$$x_n = \frac{n^2 + 3n}{3 + n^2}.$$

Solution.

$$\begin{aligned} x_n &= \frac{\frac{n^2}{3} + \frac{3n}{n^2}}{\frac{n^2}{n^2} + \frac{3}{n^2}} \\ &= \frac{1 + \frac{3}{n}}{\frac{3}{n^2} + 1} \end{aligned}$$

But

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right) &= \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{3}{n} \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{3}{n^2} + 1\right) &= \lim_{n \rightarrow \infty} \frac{3}{n^2} + \lim_{n \rightarrow \infty} 1 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)}{\lim_{n \rightarrow \infty} \left(\frac{3}{n^2} + 1\right)} \\ &= 1. \end{aligned}$$

17. Definition.

A sequence  $(a_n)$  is *bounded* if there exists a number  $M$  such that  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ . A sequence which is not bounded is called *unbounded*.

18. Example.

Consider the sequence  $a_n = \frac{n}{n+1}$ . Since  $|a_n| \leq 1$  for all  $n \in \mathbb{N}$ , this sequence is bounded.

19. Claim.

A convergent sequence is bounded.

Proof. Let  $(a_n)$  be a convergent sequence with limit  $L$ . Then there exists a number  $N$  such the open interval  $(L-1, L+1)$  contains all terms  $a_n$  with  $n > N$ . It follows that all terms of  $a_n$  are less than or equal in absolute value to the maximum  $M$  of the set

$$\{|a_1|, |a_2|, \dots, |a_N|, |L+1|, |L-1|\}$$

(Exercise: Find a counterexample to show that the converse of this result is false.)

20. Corollary.

The sequence  $a_n = n$  has no limit.

Proof. Notice that the sequence  $(a_n)$  is unbounded. If it had a limit it would be bounded.

21. Example.

Consider the sequence  $(a_n)$  where

$$a_n = 3^n.$$

Then

$$\begin{aligned} 3^n &= (1+2)^n \\ &\geq 1+2n \\ &\geq n \end{aligned}$$

It follows that the sequence  $(a_n)$  is unbounded and hence has no limit.

22. Sandwich Theorem.

Let  $(a_n)$ ,  $(b_n)$  and  $(c_n)$  be sequences. If  $a_n \leq b_n \leq c_n$  for all  $n \in \mathbb{N}$  and the sequences  $(a_n)$  and  $(c_n)$  have the same limit  $L$  then  $\lim_{n \rightarrow \infty} b_n = L$

23. Example.

$$\text{Show } \lim_{n \rightarrow \infty} \left| \frac{\cos n}{n} \right| = 0$$

Solution.

Notice that for all  $n \geq 1$

$$0 \leq \left| \frac{\cos n}{n} \right| \leq \frac{1}{n}$$

The result follows by applying the sandwich theorem.

24. Example.

Show that  $\lim_{n \rightarrow \infty} \frac{n^4}{5^n} = 0$ .

Solution.

By the binomial theorem

$$\begin{aligned} 5^n &= (1 + 4)^n \\ &\geq \binom{n}{5} 4^5 \end{aligned}$$

Let  $p(n) = \binom{n}{5} 4^5$ . Then

$$0 \leq \frac{n^4}{5^n} \leq \frac{n^4}{p(n)}$$

Since  $p$  is a polynomial of degree 5, the right hand side converges to 0 and the result then follows from the sandwich theorem.

25. Claim.

The following are equivalent:

(a)  $\lim_{n \rightarrow \infty} a_n = 0$ .

(b)  $\lim_{n \rightarrow \infty} |a_n| = 0$ .

Proof. Exercise

Hint: Notice that

$$d(|a_n|, 0) = d(a_n, 0)$$

26. Corollary.

$$\lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0.$$

27. Lemma (Bernoulli's Inequality).

If  $x > -1$  then

$$(1 + x)^n \geq (1 + nx)$$

Proof. By induction. (Exercise).

28. Claim.

If  $a > 1$  then  $a^n$  is unbounded above.

Proof. If  $a > 1$  then  $a = 1 + x$  where

$x = a - 1$ . Notice  $x > 0$ . By Bernoulli's inequality

$$a^n = (1 + x)^n > 1 + nx$$

Since  $\mathbb{N}$  is unbounded above, given  $M$  arbitrary, there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $nx > M - 1$ . Hence, for all  $n > N$   $a^n > M$  proving that the sequence  $(a^n)$  is unbounded above.

29. Claim.

If  $0 < |x| < 1$  then

$$\lim_{n \rightarrow \infty} |x|^n = 0 \quad (2)$$

Proof.

If  $0 < |x| < 1$ , there exists  $a > 0$  such that

$$|x| = \frac{1}{1 + a}$$

Hence, using Bernoulli's inequality

$$\begin{aligned} 0 \leq |x|^n &= \frac{1}{(1 + a)^n} \\ &\leq \frac{1}{1 + na}. \end{aligned}$$

Equation (2) now follows from the sandwich theorem.

30. Corollary.

If  $|x| < 1$  then

$$\lim_{n \rightarrow \infty} x^n = 0$$

31. Example.

Show that the sequence  $a_n = 3^{1/n}$  converges.

Solution.

By Bernoulli's inequality

$$\begin{aligned} \left(1 + \frac{2}{n}\right)^n &\geq 1 + n \cdot \frac{2}{n} \\ &= 3 \end{aligned}$$

and so

$$3^{1/n} \leq 1 + \frac{2}{n}$$

Again, by Bernoulli's inequality,

$$\begin{aligned} \left(1 - \frac{2}{3n}\right)^n &\geq 1 - n \cdot \frac{2}{3n} \\ &= \frac{1}{3} \end{aligned}$$

and so

$$3^{1/n} \geq \frac{1}{1 - \frac{2}{3n}}$$

It follows by the sandwich theorem that

$$\lim_{n \rightarrow \infty} 3^{1/n} = 1.$$

32. Claim.

If  $a > 0$  then  $\lim_{n \rightarrow \infty} a^{1/n} = 1$ .

Proof. (Exercise)

(Hint: If  $1 < a$  see the previous example. If  $0 < a < 1$  consider the reciprocal).

33. Claim.

$$\lim_{n \rightarrow \infty} n^{1/n} = 1. \quad (3)$$

Proof.

Notice (exercise) that  $n^{1/n} > 1$ . Hence, using the binomial theorem,

$$\begin{aligned} n &= [n^{1/n}]^n \\ &= [1 + (n^{1/n} - 1)]^n \\ &\geq \binom{n}{2} (n^{1/n} - 1)^2 \end{aligned}$$

Therefore,

$$0 \leq (n^{1/n} - 1) \leq \sqrt{\frac{2}{n-1}}$$

and (3) now follows from the sandwich theorem.

34. Definition.

A sequence  $(x_n)$  in  $R$  is:

(a) *increasing* if for all  $n, m \in \mathbb{N}$

$$n < m \rightarrow x_n < x_m$$

(b) *non-decreasing* if for all  $n, m \in \mathbb{N}$

$$n \leq m \rightarrow x_n \leq x_m$$

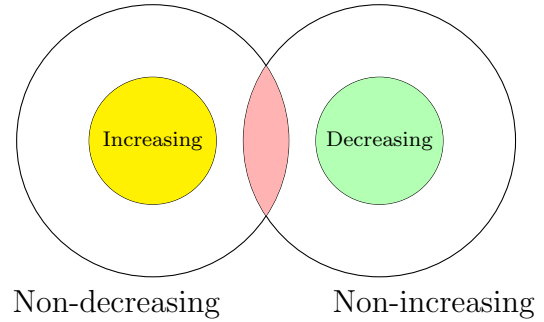
(c) *decreasing* if for all  $n, m \in \mathbb{N}$

$$n < m \rightarrow x_n > x_m$$

(d) *non-increasing* if for all  $n, m \in \mathbb{N}$

$$n \leq m \rightarrow x_n \geq x_m$$

The following Venn diagram, expresses the relationships between these concepts:



35. Dedekind (Completeness) Axiom.

Every non-empty subset of  $R$  which is bounded above has a supremum (least upper bound).

36. Claim.

A sequence  $\alpha = (a_n)$  of real numbers which is both non-decreasing bounded converges to the least upper bound of its set of terms.

Proof. Let  $L = \sup\{a_n : n \in \mathbb{N}\}$  and  $\epsilon > 0$ . Then there exists  $N$  such that

$$L - \epsilon < a_N \leq L$$

(otherwise  $L - \epsilon$  would be a smaller bound). Since  $\alpha$  is non-decreasing

$$n \geq N \rightarrow L - \epsilon < a_n \leq L$$

Hence  $\alpha$  converges to  $L$ .

37. Corollary.

A non-decreasing sequence  $\alpha = (a_n)$  of real numbers either converges or is unbounded above.

38. Definition.

Let  $(x_n)$  be a real sequence. A natural number  $m$  is called:

- (a) A *peak point* of  $(x_n)$  if  $x_m \geq x_n$  for all  $n \geq m$ .
- (b) A *p-blocker* if  $m > p$  and  $x_m \geq x_p$

39. Bolzano Weierstrass Theorem [BW].

A bounded sequence has a convergent subsequence.

Proof. (See Spivak).

Let  $m_1 < m_2 < \dots$ , be the set of peak points. If this set is infinite then the bounded non increasing subsequence

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \dots$$

must converge (to its infimum). If, on the other hand, the set of peak points is finite, there exists a last peak point  $m$  in which case there exists a convergent subsequence of  $m$ -blockers defined recursively as follows: let  $q_1 = m + 1$  and, for each  $k$ , let  $q_{k+1}$  be the minimum blocker of  $q_k$ . Then the subsequence

$$x_{q_1} \leq x_{q_2} \leq x_{q_3} \dots$$

is non decreasing and bounded below so converges.

40. Exercise.

In the proof of BW above, explain why the minimum at each stage exists.

41. Definition.

A real sequence  $(a_n)$  is *Cauchy* if for every  $\epsilon > 0$  there exists  $N$  such that

$$n, m > N \rightarrow |a_n - a_m| < \epsilon$$

.

42. Claim.

Cauchy sequences are bounded.

Proof. Let  $\epsilon > 0$ . There exists  $N$  such that for  $m, n \geq N$ ,  $|a_m - a_n| < \epsilon$ . Then, by the triangle inequality

$$|a_m| - |a_n| \leq |a_m - a_n| < \epsilon$$

for all  $m, n \geq N$ . Taking  $n = N$ . Then

$$|a_m| - |a_N| < \epsilon$$

for all  $m \geq N$ . It follows easily that  $(a_m)$  is bounded by

$$\pm \max \{|a_0|, |a_1|, \dots, |a_{N-1}|, |a_N|, \epsilon + |a_N|\}$$

43. Claim [Uses BW]. Every Cauchy sequence in  $R$  converges.

Proof. Let  $(a_n)$  be a Cauchy sequence of real numbers. Then  $(a_n)$  is bounded so by BW has a convergent subsequence  $(a_{n_k})$ . Let  $L$  be the limit of this subsequence. Then there exist  $N$  such that

$$n_k > N \rightarrow |a_{n_k} - L| < \epsilon/2$$

and there exists  $M$  such that

$$n, n_k > M \rightarrow |a_n - a_{n_k}| < \epsilon/2$$

Then, for all  $n > \max\{N, M\}$

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \epsilon$$

44. Corollary.

Dedekind completeness implies Cauchy completeness