Notes on Implication

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1. Definition .

   Let F and G be statements. The statement

   $$F \text{ implies } G$$

   also written

   $$F \Rightarrow G$$

   is said to be true if it cannot happen that F be true and at the same time G be false.

2. Corollary.

   $F \Rightarrow G$ is true if and only if G is true whenever F is.

3. Definition.

   In an implicatiom $F \implies G$, the statement F is called the *antecedent* and G is called the *consequent*.

4. Example.

   Fire is a chemical reaction involving oxygen. Thus it cannot happen that there is fire without oxygen. Hence the implication "Fire implies the presence of oxygen" is a true statement.

5. Example.

   It can (and fortunately usually does) happen that presence of oxygen can occur with no fire and so by corollary [2] above the implication

   "oxygen implies fire"

   is false. Thus order is important.

   $$F \Rightarrow G$$

   and

   $$G \Rightarrow F$$

   are NOT equivalent.

6. Example.

   The implication.

   If $2 + 2 = 5$, then cows can fly.

   is true since the left hand side is false. (It cannot therefore happen that the antecedent be true and at the same time, the consequent false.)

7. Remark.

   Implication expresses the notion of inclusion. Stating that "fire implies presence of oxygen" is equivalent to asserting that the set of situations involving fire is a subset of the set of situations in which there is presence of oxygen.

8. Example.

   The following are equivalent ways of expressing the truth of the implication

   "Fire implies the presence of oxygen"

   (a) **If** there is fire **then** there is presence of oxygen
   (b) There is **no** fire **without** oxygen.
   (c) **Whenever** there is fire there is oxygen.
   (d) Fire is a **sufficient** condition for presence of oxygen.
   (e) There is fire **only if** there is oxygen.
   (f) Oxygen is **necessary** for fire.
   (g) There is oxygen **whenever** there is fire.
   (h) There is oxygen **if** there is fire.

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(i) Every fire occurs in the presence of oxygen.

9. Claim.

   Let F and G be propositions. The following expressions are equivalent to $F \Rightarrow G$.

   (a) F **implies** G.

   (b) **If** F **then** G.

   (c) **If** F, G.

   (d) F **only if** G.

   (e) **Whenever** F, G.

   (f) **Provided that** F, G.

   (g) **Since** F, G.

10. Remark.

    Changing the position of a word can reverse an implication. The following expressions are also equivalent to $F \Rightarrow G$.

    (a) G **whenever** F.

    (b) G **provided that** F

    (c) G **if** F

    (d) G **since** F.

11. Notice that addition of the word "only" can reverse the meaning of an implication.

    F **if** G

    expresses the implication $G \implies F$ whereas

    F **only if** G

    means $F \implies G$.

12. Definition.

    Let F and G be statements and let $\neg G$ be the negation of G. The *contrapositive* of the implication $F \Rightarrow G$ is the implication $\neg G \Rightarrow \neg F$.

13. Claim.

    A statement is equivalent to its contrapositive.

    Proof.

    Using the definition of implication, the contrapositive $\neg G \Rightarrow \neg F$ is true if and only if it cannot happen that $\neg G$ is true and $\neg F$ is false. From the definition of negation, this means that it cannot happen that F is true and G is false. But this is the same as saying that $F \Rightarrow G$ is true. Therefore the statements $\neg G \Rightarrow \neg F$ and $F \Rightarrow G$ are equivalent.

14. Remark.

    The word "unless" expresses an implication. By definition,

    F **unless** G

    is equivalent to

    $$\neg G \implies F.$$

15. Corollary.

    By contraposition, and the property of double negation, F **unless** G is also equivalent to $\neg F \implies G$.

16. Example.

    Express the statement

    "There is no fire unless there is presence of oxygen" as an implication.

    Solution.

    Let F be the proposition "there is fire" and let G be "there is presence of ozygen".

    The given statement

    No F **unless** G

    is equivalent to

    $$\neg G \implies \neg F$$

which by contraposition, is in turn equivalent to the implication

$$F \implies G$$

Thus, the given statement is equivalent to the implication

"If there is fire, then there is oxygen present."

17. Definition.

    If an implication $F \implies G$ is true we say that

    (a) F is **sufficient** for G and
    (b) G is **necessary** for F .

    If an implication $F \implies G$ is false we say that

    (a) F is **not sufficient** for G and
    (b) G is **not necessary** for F .

    This, a **sufficient condition** is the antecedant of a true implication. A **necessary condition** is the consequent of a true implication.

18. Example.

    The following statements both express the falsehood of the statement

    "presence of oxygen implies fire"

    (a) "Presence of oxygen is not a sufficient condition for fire".
    (b) "Fire is not a necessary condition for presence of oxygen".

19. Example.

    Let D be the assertion

    "The function is differentiable"

    and C the assertion

    "The function is continuous."

    Suppose that the implication

    "C only if D"

    is false. Which of the following expressions must be false?

    (a) D if C
    (b) C is sufficient for D
    (c) D is necessary for C
    (d) $\neg(C \wedge \neg D)$
    (e) C is not sufficient for D
    (f) D is not necessary for C
    (g) $D \Rightarrow C$

    Solution.

    It is given that the implication $C \Rightarrow D$ is false. Hence any statement which is equivalent (i.e., with the same truth value) to $C \Rightarrow D$ must also be false. Items (a),(b),(c),(d) are all equivalent ways of saying (in the metalanguage English) that $C \Rightarrow D$ and hence are false.

    Remark.

    Notice that statement (d), in the previous example, is a way of expressing implication in terms of negation ($\neg$) and conjunction ($\wedge$). An implication $C \implies D$ is true if and only if it CANNOT happen (negation) that the antecedent C is true AND, at the same time, the consequent D false. Hence the statements $\neg(C \wedge \neg D)$ and $C \implies D$ are equivalent. I.e.,

    $$\neg(C \wedge \neg D) \iff (C \implies D).$$

Exercises

1. Let F be the statement "It is raining" and G is the statement "I use my umbrella". Write the following implications symbolically.

   (a) "I use my umbrella whenever it rains".

   (b) "When it rains I use my umbrella".

   (c) "I use my umbrella if it rains".

   (d) It rains only if I use my umbrella.

   (e) I use my umbrella only if it rains.

   (f) If it rains it is necessary that I use my umbrella.

   (g) Rain is sufficient for me to use my umbrella.

2. Write 10 different statements which are equivalent to

   "There is no smoke without fire".

3. Suppose that an implication $F \Rightarrow G$ is true. If G is false, what can be deduced about the truth of F? Explain.

4. Let n be a natural number.

   (a) Write the contrapositive of the statement "If 2 is a factor of n^2 then 2 is a factor of n".

   (b) Prove the contrapositive of the statement in part (a).

5. Let $p(x)$ and $q(x)$ be the predicates:

   $$p(x) : x^2 - 3x + 2 = 0$$
   $$q(x) : (x-1)^3(x-2)(x-3)x = 0$$

   Show that: $\forall x \in \mathbb{R}, \quad p(x) \implies q(x)$