

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

SETS – SAMPLE SOLUTIONS AND EXERCISES

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1. If $A = \{1, 2, \{3\}\}$, which of the following are true?

(a) $\{3\} \subseteq A$

False

(b) $3 \in A$

False

(c) $\{1, 2, 3\} \subseteq A$

False

(d) $\{3\} \in A$

True

2. Find the elements of the following sets (say whether they are empty).

(a) $\{x \in \mathbb{N} : 1 \leq x^2 \leq 100 \text{ and } x \text{ is divisible by } 3\}$

Solution. $\{3, 6, 9\}$.

(b) $\{n \in \mathbb{N} : 2 < n/6 < 3\}$

Solution. $\{13, 14, 15, 16, 17\}$.

(c) $\{x \in \mathbb{N} : x^2 - x = 6\}$

Solution.

$$x^2 - x - 6 = 0 \iff (x - 3)(x + 2) = 0.$$

Since $x \in \mathbb{N}$, it must be that $x = 3$. Hence

$$\{x \in \mathbb{N} : x^2 - x = 6\} = \{3\}.$$

3. Find the elements of the following sets (say whether they are empty).

(a) $\{x \in \mathbb{Z} : |x| < 4\}$

Solution.

$$\begin{aligned} |x| < 4 &\iff d(x, 0) < 4 \\ &\iff -4 < x < 4. \end{aligned}$$

Hence $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

(b) $\{x \in \mathbb{Z} : x^{2020} < 0\}$

Solution.

$$x^{2020} = (x^{1010})^2 \geq 0,$$

and the inequality is strict only if the square is negative, which never occurs. Hence the set is empty.

¹<https://pennance.us>

4. Let $A = \{x \in \mathbb{R} : 4 \leq x \leq 8\}$ and $B = \{x \in \mathbb{R} : x \geq 6\}$. Suppose A' and B' denote the complements of the sets A and B . Write the following sets as an interval or a union of intervals.

$$(a) \ A \cup B \qquad | \qquad (b) \ A' \cup B' \qquad | \qquad (c) \ \mathbb{R} \setminus (A \cap B)$$

Solution.

$$(a) \ [4, \infty) \qquad | \qquad (b) \ (-\infty, 6) \cup (8, \infty) \qquad | \qquad (c) \ (-\infty, 6) \cup (8, \infty)$$

5. Let

$$E = \{k \in \mathbb{N} : 6 \leq k \leq 6671\}, \quad A = \{k \in E : k^2 = 49 \vee \sqrt{k} = 9\}, \quad B = \{a \in A : a \neq 7\}.$$

Find:

$$\begin{array}{l|l} (a) \ A \setminus B & (c) \text{ the number of elements in } E \\ (b) \ A \times B & \end{array}$$

Solution.

$$(a) \ A = \{7, 81\} \text{ and } B = \{81\}. \text{ Hence } A \setminus B = \{7\}.$$

$$(b) \ A \times B = \{(7, 81), (81, 81)\}.$$

$$(c) \ E = \{6, 7, 8, \dots, 6671\}. \text{ Let } |E| \text{ denote the number of elements of } E. \text{ Subtracting a constant from each element is a bijection and so does not change the number of elements. Hence}$$

$$\begin{aligned} |E| &= |\{6, 7, 8, \dots, 6671\}| \\ &= |\{6 - 5, 7 - 5, 8 - 5, \dots, 6671 - 5\}| \\ &= |\{1, 2, 3, \dots, 6666\}| \\ &= 6666. \end{aligned}$$

6. In a school of 120 students, 60 take the English class and 80 take the Spanish class. There are 95 students taking at least one of the two courses.

- Draw a Venn diagram.
- How many students take neither course?
- How many students take both courses?
- How many students take exactly one of the two courses?

Solution. Let I be the set of students taking English and E the set of students taking Spanish.

$$\begin{aligned} |I \cup E| &= |I| + |E| - |I \cap E| \\ 95 &= 60 + 80 - |I \cap E| \\ |I \cap E| &= 45. \end{aligned}$$

Thus 45 students take both courses. It follows that 15 take only English and 35 take only Spanish. Hence 50 take exactly one course, and $120 - 95 = 25$ take neither course.

7. In a group of 1234 students, 600 took calculus and 700 took chemistry. Find:

- | | |
|---|---|
| (a) the minimum possible number of students who could have taken both courses. | (b) the maximum possible number of students who could have taken both courses. |
|---|---|

Solution. $600 + 700 - 1234 = 66$.

Solution. $\min(600, 700) = 600$.

8. Let

$$E = \{x \in \mathbb{R} : 0 \leq x \leq 12\}, \quad A = \{x \in E : 2 < x < 10\}, \quad B = \{x \in A : 5 \leq x\}.$$

Find:

- | | |
|--|--|
| (a) $E \setminus (A \cup B)$
(b) $(E \setminus A) \cap (E \setminus B)$ | (c) $E \setminus (A \cap B)$
(d) $(E \setminus A) \cup (E \setminus B)$ |
|--|--|

Solution. In interval notation,

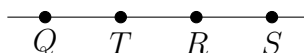
$$\begin{aligned} E &= [0, 12], \\ A &= (2, 10), \\ B &= [5, 10), \\ A \cup B &= (2, 10), \\ A \cap B &= [5, 10). \end{aligned}$$

Hence

$$\begin{aligned} E \setminus (A \cup B) &= [0, 2] \cup [10, 12] = (E \setminus A) \cap (E \setminus B), \\ E \setminus (A \cap B) &= [0, 5] \cup [10, 12] = (E \setminus A) \cup (E \setminus B). \end{aligned}$$

Additional exercise: Explain why the answers to (a) and (b) are the same, and why the answers to (c) and (d) are the same.

Recall that $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. Since the union of two sets is defined using “ $\vee$ ” and the intersection using “ $\wedge$ ”, the complement of the union of two sets is the intersection of the complements. Similarly, the complement of the intersection of two sets is the union of the complements. These facts are called De Morgan’s laws.

9. Consider the points in the diagram:  If $d(Q, R) = d(T, S)$, prove that

$$d(Q, T) = d(R, S).$$

Solution. Notice that $[Q, R] = [Q, T] \cup (T, R]$ and $[Q, T] \cap (T, R] = \emptyset$. Since the intervals $[Q, T]$ and $(T, R]$ are disjoint,

$$d(Q, R) = d(Q, T) + d(T, R).$$

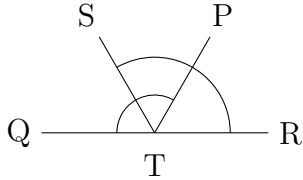
By a similar argument,

$$d(T, S) = d(T, R) + d(R, S).$$

Hence

$$\begin{aligned}
 d(Q, R) &= d(T, S) \\
 \implies d(Q, T) + d(T, R) &= d(T, R) + d(R, S) \\
 \implies d(Q, T) &= d(R, S).
 \end{aligned}$$

10. Consider the angles in the diagram. If $\angle QTP = \angle RTS$, prove that $\angle QTS = \angle RTP$.



Solution. Since the angles $\angle QTS$ and $\angle STP$ are disjoint and concatenate to $\angle QTP$,

$$\angle QTP = \angle QTS + \angle STP.$$

By a similar argument,

$$\angle RTS = \angle RTP + \angle STP.$$

Hence

$$\begin{aligned}
 \angle QTP &= \angle RTS \\
 \implies \angle QTS + \angle STP &= \angle RTP + \angle STP \\
 \implies \angle QTS &= \angle RTP.
 \end{aligned}$$

11. Let A and B be finite sets. Suppose $|A|$ denotes the number of elements in A . If $|A| = |B|$, prove that

$$|B \setminus A| = |A \setminus B|.$$

Solution. Notice (draw a Venn diagram) that A is the disjoint union of $A \setminus B$ and $A \cap B$.

Hence

$$|A| = |A \setminus B| + |A \cap B|.$$

By a similar argument,

$$|B| = |B \setminus A| + |A \cap B|.$$

Hence

$$\begin{aligned}
 |A| &= |B| \\
 \implies |A \setminus B| + |A \cap B| &= |B \setminus A| + |A \cap B| \\
 \implies |A \setminus B| &= |B \setminus A|.
 \end{aligned}$$

Remark: Problems 9, 10, and 11 are solved by an identical technique. Look for what they have in common.