

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

SEQUENCES – SAMPLE SOLUTIONS AND EXERCISES.

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1. Express in summation notation the partial sum S_n of the geometric sequence

$$4, \frac{4}{10}, \frac{4}{100}, \dots$$

Express S_{12} as a decimal (exact value required). **Solution.**

$$\begin{aligned} S_{12} &= \sum_{i=1}^{12} 4 \left(\frac{1}{10} \right)^{i-1} \\ &= 4 \cdot \frac{1 - (1/10)^{12}}{1 - 1/10} \\ &= \underbrace{4.444444 \dots 4}_{12 \text{ fours}}. \end{aligned}$$

The last line follows, without further computation, from the definition of decimal representation.

2. Let (t_n) be a geometric sequence with $t_1 = 16$ and

$$t_{n+1} = \frac{t_n}{10}, \quad n \geq 1.$$

- (a) Write an explicit formula for the term t_n . **Solution.**

$$t_n = \frac{16}{10^{n-1}}.$$

- (b) Find $\sum_{i=1}^{\infty} t_i$. **Solution.**

$$S_{\infty} = \frac{16}{1 - 1/10} = \frac{160}{9}.$$

3. (a) Write $0.0\overline{111}$ in sigma notation. **Solution.**

$$0.0\overline{111} = \frac{1}{100} \sum_{i=0}^{\infty} \left(\frac{1}{10} \right)^i.$$

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- (b) Write $0.0\overline{111}$ in the form n/m where $n, m \in \mathbb{N}$. **Solution.** By the formula for an infinite geometric series,

$$0.0\overline{111} = \frac{1/100}{1 - 1/10} = \frac{1}{90}.$$

4. Write

$$-36 + 25 - 16 + 9 - 4 + 1$$

in Σ notation. **Solution.**

$$-36 + 25 - 16 + 9 - 4 + 1 = \sum_{k=1}^6 (-1)^k (7 - k)^2.$$

There are infinitely many correct answers. Another is

$$\sum_{k=1}^6 (-1)^{k+1} (7 - k)^2 \cdot (-1) = \sum_{j=0}^5 (-1)^{j+1} (6 - j)^2.$$

5. Let (t_n) be the arithmetic sequence given by $t_{n+1} = 5 + t_n$ and $t_1 = 15$. Find $\sum_{n=26}^{141} t_n$.

Solution. The terms are given explicitly by

$$t_n = 15 + (n - 1)5.$$

Hence

$$\begin{aligned} t_1 &= 15, \\ t_{141} &= 15 + 140 \cdot 5 = 715, \\ t_{25} &= 15 + 24 \cdot 5 = 135. \end{aligned}$$

If S_n denotes the sum of the first n terms, then

$$S_n = n \cdot \frac{t_1 + t_n}{2}.$$

$$\begin{aligned} S_{141} &= 141 \cdot \frac{15 + 715}{2} = 51465, \\ S_{25} &= 25 \cdot \frac{15 + 135}{2} = 1875. \end{aligned}$$

Therefore

$$\sum_{n=26}^{141} t_n = S_{141} - S_{25} = 51465 - 1875 = 49590.$$

6. Let (t_n) be an arithmetic sequence. If $t_{35} = 100$ and $t_{15} = 40$, find:

- (a) the term t_{105} ; **Solution.** Let t_1 be the first term and $d = t_{n+1} - t_n$ the common difference. Then $t_n = t_1 + (n - 1)d$. Putting $n = 15$ and $n = 35$ yields

$$\begin{aligned}40 &= t_1 + 14d, \\100 &= t_1 + 34d.\end{aligned}$$

Solving gives $t_1 = -2$ and $d = 3$, so

$$t_n = 3n - 5.$$

In particular, $t_{105} = 310$.

- (b) the sum $\sum_{n=26}^{135} t_n$. **Solution.** If S_n is the sum of the first n terms, then

$$S_n = n \cdot \frac{t_1 + t_n}{2}.$$

Now $t_{135} = 400$ and $t_{25} = 70$, so

$$\begin{aligned}S_{135} &= 135 \cdot \frac{-2 + 400}{2} = 135 \cdot 199 = 26865, \\S_{25} &= 25 \cdot \frac{-2 + 70}{2} = 850.\end{aligned}$$

Hence

$$\sum_{n=26}^{135} t_n = S_{135} - S_{25} = 26865 - 850 = 26015.$$

7. If $x = 1/10$, find in decimal form the sum of the geometric series $\sum_{i=3}^{\infty} x^i$ (do not use a calculator). **Solution.**

$$\sum_{i=3}^{\infty} x^i = x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1 - x} = 10^{-3} \cdot \frac{1}{9/10} = 10^{-3} \cdot 1.\bar{1} = 0.00\bar{1}.$$

8. If $x = 100$, write in decimal form the sum of the finite geometric series

$$1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \frac{16}{x^4}.$$

Solution.

$$1 + 0.02 + 0.0004 + 0.000008 + 0.00000016 = 1.02040816.$$

9. If $x = 1/100$, find $\sum_{i=2}^6 x^i$ in decimal form. Do not use a calculator. **Solution.** The series is geometric and $x = 10^{-2}$:

$$\begin{aligned}\sum_{i=2}^6 x^i &= x^2 + x^3 + x^4 + x^5 + x^6 = x^2(1 + x + x^2 + x^3 + x^4) \\ &= 10^{-4}(1 + 0.01 + 0.0001 + 0.000001 + 0.00000001) \\ &= 10^{-4}(1.01010101) = 0.000101010101.\end{aligned}$$

10. If $x = 1/10$, find in decimal form the sum

$$x^2 + x^3 + \cdots + x^{10}$$

(do not use a calculator). **Solution.** The series is geometric and $x = 10^{-1}$:

$$\begin{aligned}\sum_{i=2}^{10} x^i &= x^2(1 + x + \cdots + x^8) = 10^{-2}(1 + 0.1 + 0.01 + \cdots + 10^{-8}) \\ &= 10^{-2}(1.11111111) = 0.0111111111.\end{aligned}$$

11. (a) Define *arithmetic sequence*. **Solution.** A sequence f with terms $f(1), f(2), \dots, f(n), \dots$ is arithmetic if there is a constant d such that

$$f(n+1) - f(n) = d \quad \text{for all } n \geq 1.$$

- (b) Let $f(k) = 4k + 5$ for $k \in \{1, 2, \dots\}$. Find the number d such that

$$f(10^7 + 4) = f(10^7 + 3) + d.$$

Solution. For any natural number k ,

$$f(k+1) - f(k) = 4(k+1) + 5 - (4k + 5) = 4,$$

which is independent of k . Hence $d = 4$.

12. Let $f(k) = 4k + 5$ for $k \in \{1, 2, \dots\}$. Find $\sum_{i=1}^{20} f(i)$. **Solution.**

$$\sum_{i=1}^{20} f(i) = 20 \cdot \frac{f(1) + f(20)}{2} = 20 \cdot \frac{9 + 85}{2} = 940.$$

13. If $f(k) = 4k$, find $\sum_{i=1}^{20} [f(i+1) - f(i)]$. **Solution.** The sum is telescoping:

$$\sum_{i=1}^{20} [f(i+1) - f(i)] = f(21) - f(1) = 84 - 4 = 80.$$

14. Find x if the geometric series on the right-hand side of

$$\frac{x+2}{2x} = x + x^2 + x^3 + \cdots$$

is convergent. **Solution.**

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

provided $|x| < 1$ and $x \neq 0$. Hence

$$\begin{aligned} \frac{x+2}{2x} &= \frac{x}{1-x} \\ (x+2)(1-x) &= 2x^2 \\ 3x^2 + x - 2 &= 0 \\ (x+1)(3x-2) &= 0 \\ x &\in \left\{-1, \frac{2}{3}\right\}. \end{aligned}$$

An infinite geometric series converges if and only if the common ratio is strictly less than 1 in absolute value. Thus the only possibility is $x = 2/3$. It is left as an exercise to verify that $x = 2/3$ does satisfy the original equation.

15. A ball has the property that, each time it is dropped, it rebounds to one third of the height of the previous bounce. Find the height from which the ball should be dropped so that it travels a total distance of 60 feet. **Solution.**

$$\begin{aligned} 60 &= h + 2\frac{h}{3} + 2\frac{h}{3^2} + 2\frac{h}{3^3} + \cdots \\ &= \left(2h + 2\frac{h}{3} + 2\frac{h}{3^2} + \cdots\right) - h \\ &= 2h\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right) - h \\ &= 2h \cdot \frac{1}{1 - 1/3} - h \\ &= 2h. \end{aligned}$$

Hence $h = 30$ ft.

16. Write

$$-\frac{1}{23} + \frac{2}{25} - \frac{3}{27} + \frac{4}{29} - \frac{5}{31}$$

in Σ notation. **Solution.**

$$\sum_{i=1}^5 (-1)^i \frac{i}{2i+21}.$$

There are infinitely many correct solutions.

17. Write

$$-\frac{1}{13} + \frac{2}{15} - \frac{3}{17} + \frac{4}{19} - \frac{5}{21}$$

in Σ notation. **Solution.**

$$\sum_{i=1}^5 (-1)^i \frac{i}{2i+11}.$$

There are infinitely many correct solutions.