

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE QUESTIONS ON THE REAL NUMBERS

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____
Student #: _____ Professor: Philip Pennance Section: _____

1. Mention the axioms which justify each of the following:

- (a) $x + 0 = x$ Property of zero (additive identity).
- (b) $(x^2 + y) + 2z = 2z + (x^2 + y)$ Commutativity of addition.
- (c) $a + c + (-c) = a + 0$ Property of the opposite (additive inverse).
- (d) $x^2 + x^3 = x^2(1 + x)$ Distributivity.
- (e) $x^5 = 1 \cdot x^5$ Property of the multiplicative identity.

2. Mention the definitions which justify each of the following:

- (a) $5x - (-2x) = 5x + 2x$
- (b) $11/b = (11)(1/b)$
- (c) $x/y = x(1/y)$
- (d) $(1/3)x = x/3$
- (e) $3a - 7a = 3a + (-7a)$

3. Find the exact value of $|3.\overline{99} - 4|$.

Solution. Let $x = 3.\overline{99}$. Then

$$10x = 39.\overline{99}, \quad 10x - x = 36, \quad 9x = 36, \quad x = 4.$$

Alternatively,

$$3.\overline{99} = 3 + \sum_{n=1}^{\infty} \frac{9}{10^n} = 3 + \frac{9/10}{1 - 1/10} = 3 + 1 = 4.$$

Hence $|3.\overline{99} - 4| = |4 - 4| = 0$. Bottom line: The number $3.\overline{99}$ is exactly 4, so the absolute value is 0, not a positive infinitesimal.

4. Explain carefully, with reference to the appropriate definitions, why each of the following formulae is not true for all real numbers.

- (a) $\sqrt{4} = \pm 2$
- (b) $\sqrt{x^2} = \pm x$
- (c) $\sqrt{x^2} = x$
- (d) $\sqrt{x^2} = -x$

Solution. By definition, if $a \geq 0$, then $\sqrt{a}$ is the unique nonnegative real number whose square is a . The symbol $\sqrt{\cdot}$ therefore denotes a single value, never a pair of values.

- (a) The equation $t^2 = 4$ has two solutions, $t = \pm 2$. But $\sqrt{4}$ is the principal square root, so $\sqrt{4} = 2$, not ± 2 .
 - (b) $\pm x$ is not a well-defined real number (it names two candidates). The correct identity is $\sqrt{x^2} = |x|$, which is never negative.
 - (c) If $x = -3$, then $\sqrt{x^2} = \sqrt{9} = 3 \neq -3$. So $\sqrt{x^2} = x$ fails for every negative x .
 - (d) If $x = 3$, then $\sqrt{x^2} = \sqrt{9} = 3 \neq -3$. So $\sqrt{x^2} = -x$ fails for every positive x . (It does hold when $x \leq 0$, since then $|x| = -x$.)
5. Solve the equation $x = -x$. Mention the axiom(s) which justify each step in your solution.

Solution.

- (a) $x = -x$ Hypothesis.
- (b) $x + x = x + x$ Reflexive property of equality.
- (c) $x + x = x + (-x)$ Substitution.
- (d) $x + (-x) = 0$ Property of the opposite (additive inverse).
- (e) $x + x = 0$ Transitive property of equality.
- (f) $1 \cdot x + 1 \cdot x = 0$ Property of one (multiplicative identity).
- (g) $(1 + 1) \cdot x = 0$ Distributivity.
- (h) $2 \cdot x = 0$ Theorem: $1 + 1 = 2$.
- (i) $\frac{1}{2} \cdot (2 \cdot x) = \frac{1}{2} \cdot (2 \cdot x)$ Reflexivity of equality.
- (j) $\frac{1}{2} \cdot (2 \cdot x) = \frac{1}{2} \cdot 0$ Substitution.
- (k) $(\frac{1}{2} \cdot 2) \cdot x = \frac{1}{2} \cdot 0$ Associativity of multiplication.
- (l) $1 \cdot x = \frac{1}{2} \cdot 0$ Property of the multiplicative inverse.
- (m) $x = \frac{1}{2} \cdot 0$ Property of the multiplicative identity.
- (n) $x = 0$ Theorem: $0 \cdot x = 0$.

Remarks. Excessive (though not all!) detail has been provided. In practice, proofs are not written in this way, and this level of detail is not required. Properties of equality are commonly used in proofs without mention.

6. Solve the equation $\frac{1}{x} = -\frac{1}{x}$. Mention the axiom(s) which justify each step in your solution.

Solution. The expression $1/x$ is defined only for $x \neq 0$. Adding $1/x$ to both sides gives

$$\frac{2}{x} = 0.$$

No real number satisfies this, because a product $2 \cdot (1/x)$ is zero only if one factor is zero, and $2 \neq 0$. Equivalently, if $x \neq 0$ one may multiply both sides by x to obtain $1 = -1$, a contradiction. Thus the equation has no real solution.

7. Solve the equation $x - 4 = 2x - 7$

Solution. (Some steps omitted.)

$$\begin{aligned} x - 4 = 2x - 7 &\stackrel{1}{\iff} 2x + (-7) = x + (-4) \\ &\stackrel{2}{\iff} (2x + (-7)) + 7 = (x + (-4)) + 7 \\ &\stackrel{3}{\iff} 2x + (-7 + 7) = x + (-4 + 7) \\ &\stackrel{4}{\iff} 2x + 0 = x + 3 \\ &\stackrel{5}{\iff} 2x = x + 3 \\ &\stackrel{6}{\iff} (-x) + 2x = (-x) + x + 3 \\ &\stackrel{7}{\iff} x = 3. \end{aligned}$$

1. Definition of subtraction

3. Associativity of addition

6. Addition of the additive inverse

2. Addition of the additive inverse

4. Additive inverse

verse

5. Additive identity

7. Additive inverse and identity

8. Solve the equation $x^2 - 3x + 2 = 0$. Justify every step.
9. Suppose that $x < y$. If $(x - \pi)^2 = (y - \pi)^2 = 2$, find the point p between x and y such that $3|x - p| = 4|y - p|$.

Solution.

$$(x - \pi)^2 = 2 \implies d(x, \pi) = \sqrt{2}.$$

Similarly, $d(y, \pi) = \sqrt{2}$. It follows that π is the midpoint of x and y , and that

$$x = \pi - \sqrt{2}, \quad y = \pi + \sqrt{2}.$$

The point p lies between x and y so that the ratio $xp : py = 4 : 3$. Hence

$$\begin{aligned} p &= x + \frac{4}{7}(y - x) \\ &= \pi - \sqrt{2} + \frac{4}{7} \cdot 2\sqrt{2} \\ &= \pi + \frac{\sqrt{2}}{7}. \end{aligned}$$

10. Solve:

(a) $-(2x) = 2(-x)$

(b) $|2x| = x + 1$

(c) $|-2x - 10| - 10 = 0$

(d) $\frac{1}{x} + \frac{1}{x-3} = \frac{9}{x^2 - 3x}$

11. Find the distance between the point (a, b) and the point $(-2, 3)$ if

$$a^2 + b^2 - 4a + 6b - 1 = 0.$$

12. Use the formula

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

to show that $x^2 - x + 1 \geq 3/4$.

Solution.

$$\begin{aligned} x^2 - x + 1 &= \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 \\ &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \\ &\geq \frac{3}{4}. \end{aligned}$$

13. Find α and β such that

$$\frac{4}{3}x - \alpha x = \frac{16}{3}(x - 2) + \beta$$

for all $x \in \mathbb{R}$.

Solution. Putting $x = 0$ shows that $\beta = 32/3$. Hence

$$\frac{4}{3}x - \alpha x = \frac{16}{3}(x - 2) + \frac{32}{3}.$$

Substituting $x = 2$ yields

$$\frac{8}{3} - 2\alpha = \frac{32}{3},$$

so $\alpha = -4$.

14. A socialist mathematician proclaims “all real numbers are equal.” Verify that some real numbers are “more equal than others” by finding and correcting the error in his “proof.”

“Proof.” Let $x, y \in \mathbb{R}$ and let m be the average of x and y .

$$m = (x + y)/2 \tag{1}$$

$$x + y = 2m \tag{2}$$

$$(x + y)(x - y) = 2m(x - y) \tag{3}$$

$$x^2 - y^2 = 2mx - 2my \tag{4}$$

$$x^2 - 2mx = y^2 - 2my \tag{5}$$

$$x^2 - 2mx + m^2 = y^2 - 2my + m^2 \tag{6}$$

$$(x - m)^2 = (y - m)^2 \tag{7}$$

$$(x - m) = (y - m) \tag{8}$$

$$x = y \tag{8}$$

Solution. By the one-dimensional Pythagorean theorem,

$$d(m, x) = \sqrt{(x - m)^2} = |x - m|.$$

Hence line (7) should read $|x - m| = |y - m|$, which merely states that x and y are equidistant from the midpoint m .