

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE QUESTIONS ON QUADRATICS

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____
Student #: _____ Professor: Philip Pennance Section: _____

1. Solve $x^2 - 3x + 2 = 0$.

Solution.

$$\begin{aligned}x^2 - 3x + 2 = 0 &\iff (x - 1)(x - 2) = 0 \\&\iff x - 1 = 0 \text{ or } x - 2 = 0 \\&\iff x = 1 \text{ or } x = 2.\end{aligned}$$

Remark: It is a theorem that $ab = 0 \implies a = 0$ or $b = 0$.

2. Express each of the following as a difference of two squares.

(a) $x^2 + 3x - 8$

Solution.

$$\begin{aligned}x^2 + 3x - 8 &= \left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8 \\&= \left(x + \frac{3}{2}\right)^2 - \frac{41}{4} \\&= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{41}{4}}\right)^2.\end{aligned}$$

(b) $x^2 - 13x - 43$

Solution.

$$\begin{aligned}x^2 - 13x - 43 &= \left(x - \frac{13}{2}\right)^2 - \left(\frac{13}{2}\right)^2 - 43 \\&= \left(x - \frac{13}{2}\right)^2 - \frac{341}{4} \\&= \left(x - \frac{13}{2}\right)^2 - \left(\sqrt{\frac{341}{4}}\right)^2.\end{aligned}$$

(c) $2x^2 + \pi x - \sqrt{17}$

Solution.

$$\begin{aligned} 2x^2 + \pi x - \sqrt{17} &= 2 \left[x^2 + \frac{\pi}{2}x - \frac{\sqrt{17}}{2} \right] \\ &= 2 \left[\left(x + \frac{\pi}{4} \right)^2 - \left(\frac{\pi}{4} \right)^2 - \frac{\sqrt{17}}{2} \right]. \end{aligned}$$

Let $D = \frac{\pi^2}{16} + \frac{\sqrt{17}}{2}$. Then

$$2x^2 + \pi x - \sqrt{17} = 2 \left[\left(x + \frac{\pi}{4} \right)^2 - (\sqrt{D})^2 \right] = \left(\sqrt{2}x + \frac{\sqrt{2}\pi}{4} \right)^2 - (\sqrt{2D})^2.$$

(d) $x^2 + 4$

Solution.

$$x^2 + 4 = x^2 - (2i)^2.$$

3. Find $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$2x^2 + 3x - 12 = \alpha(x - \beta)^2 + \gamma.$$

Solution.

$$\begin{aligned} 2x^2 + 3x - 12 &= 2 \left[x^2 + \frac{3}{2}x - 6 \right] \\ &= 2 \left[\left(x + \frac{3}{4} \right)^2 - \left(\frac{3}{4} \right)^2 - 6 \right] \\ &= 2 \left(x + \frac{3}{4} \right)^2 - \frac{105}{8}. \end{aligned}$$

Thus $\alpha = 2$, $\beta = -\frac{3}{4}$, $\gamma = -\frac{105}{8}$.

4. Solve $-2x + 4 < -5x$.

Solution.

$$\begin{aligned} -2x + 4 &< -5x \\ 3x &< -4 \\ x &< -\frac{4}{3}. \end{aligned}$$

5. Show that $0 < x < y \implies \sqrt{x} < \sqrt{y}$.

Solution. Suppose $0 < x < y$ and, to the contrary, that $\sqrt{x} \geq \sqrt{y}$. Multiplying first by $\sqrt{x}$ and then by $\sqrt{y}$ gives

$$x \geq \sqrt{x}\sqrt{y} \quad \text{and} \quad \sqrt{x}\sqrt{y} \geq y,$$

hence $x \geq y$, a contradiction. Therefore $\sqrt{x} < \sqrt{y}$.

6. Write each set as an interval or a union of intervals.

(a) $\{x \in \mathbb{R} : x^2 - x - 6 \leq 0\}$

Solution.

$$x^2 - x - 6 \leq 0 \iff (x - 3)(x + 2) \leq 0 \iff -2 \leq x \leq 3.$$

So the set is $[-2, 3]$.

(b) $\{x \in \mathbb{R} : \frac{x - 3}{x + 2} \geq 0\}$

Solution.

$$\begin{aligned} \frac{x - 3}{x + 2} \geq 0 &\iff (x - 3)(x + 2) \geq 0 \text{ and } x \neq -2 \\ &\iff x < -2 \text{ or } x \geq 3. \end{aligned}$$

So the set is $(-\infty, -2) \cup [3, \infty)$.

7. Write

$$\{x \in \mathbb{R} : x^2 - x - 6 \leq 0\} \cap \left\{x \in \mathbb{R} : \frac{1 - x}{x + 3} \geq 1\right\}$$

in interval notation.

Solution.

$$\begin{aligned} \frac{1 - x}{x + 3} \geq 1 &\iff \frac{1 - x}{x + 3} - 1 \geq 0 \\ &\iff \frac{1 - x - (x + 3)}{x + 3} \geq 0 \\ &\iff \frac{-2x - 2}{x + 3} \geq 0 \\ &\iff \frac{x + 1}{x + 3} \leq 0 \\ &\iff (x + 1)(x + 3) \leq 0 \text{ and } x \neq -3 \\ &\iff -3 < x \leq -1. \end{aligned}$$

On the other hand, $x^2 - x - 6 \leq 0 \iff x \in [-2, 3]$. The intersection is

$$(-3, -1] \cap [-2, 3] = [-2, -1].$$

8. Let $a, b \in \mathbb{R}$ with $a < b$. If $0 < t < 1$, show that

$$a < ta + (1 - t)b < b.$$

Solution. Since $a < b$ and $t > 0$, one has $ta < tb$. Since $a < b$ and $1 - t > 0$, one has $(1 - t)a < (1 - t)b$. Therefore

$$a = ta + (1 - t)a < ta + (1 - t)b < tb + (1 - t)b = b.$$

9. Factor $x^2 - \frac{3}{2}x - 6$.

Solution.

$$\begin{aligned}x^2 - \frac{3}{2}x - 6 &= \left(x - \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 6 \\&= \left(x - \frac{3}{4}\right)^2 - \frac{33}{16} \\&= \left(x - \frac{3}{4} + \sqrt{\frac{33}{16}}\right)\left(x - \frac{3}{4} - \sqrt{\frac{33}{16}}\right).\end{aligned}$$

10. Solve by completing the square. Do not use the quadratic formula.

$$2x^2 + 3x - 2 = 0.$$

11. Solve $x^2 + 4x - \sqrt{17} = 0$.

12. Solve $2x^2 - \sqrt{13} = 0$.

13. Write $y^4 + 13y^2 - 100$ as a difference of two squares.

Solution.

$$\begin{aligned}y^4 + 13y^2 - 100 &= \left(y^2 + \frac{13}{2}\right)^2 - \left(\frac{13}{2}\right)^2 - 100 \\&= \left(y^2 + \frac{13}{2}\right)^2 - \frac{569}{4} \\&= \left(y^2 + \frac{13}{2}\right)^2 - \left(\sqrt{\frac{569}{4}}\right)^2.\end{aligned}$$

14. Solve $x^2 + \sqrt{13} = 0$ over the complex numbers.

15. (a) Write $x^2 - 4x - 96$ as a difference of two squares.

- (b) Solve $x^2 - 4x - 96 = 0$.

16. Prove by completing the square that

$$2x^2 + \sqrt{13}x - 100 = 2\left(x + \frac{\sqrt{13}}{4}\right)^2 + \left[-2\left(\frac{\sqrt{13}}{4}\right)^2 - 100\right].$$

17. Express $2x^2 + \sqrt{13}x - 100$ as a difference of two squares.

18. Prove the formula

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2.$$

Solution. By the difference of squares,

$$\begin{aligned}\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 &= \left(x + \frac{b}{2} + \frac{b}{2}\right)\left(x + \frac{b}{2} - \frac{b}{2}\right) \\&= (x + b)x = x^2 + bx.\end{aligned}$$

19. Use the formula of the previous item to express $x^2 + bx + c$ as a difference of two squares. You may assume $b^2 \geq 4c$.

Solution.

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4} \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\sqrt{\frac{b^2 - 4c}{4}}\right)^2. \end{aligned}$$

20. Use the same formula to solve $x^2 + 4x - 14 = 0$.

Solution.

$$\begin{aligned} x^2 + 4x - 14 = 0 &\iff (x + 2)^2 - 4 - 14 = 0 \\ &\iff (x + 2)^2 - 18 = 0 \\ &\iff (x + 2 - 3\sqrt{2})(x + 2 + 3\sqrt{2}) = 0. \end{aligned}$$

Hence $x = -2 \pm 3\sqrt{2}$.

21. Write $x^4 - \pi x^2 + \sqrt{\pi}$ as a difference of two squares.

Solution.

$$\begin{aligned} x^4 - \pi x^2 + \sqrt{\pi} &= \left(x^2 - \frac{\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2 + \sqrt{\pi} \\ &= \left(x^2 - \frac{\pi}{2}\right)^2 - \left[\left(\frac{\pi}{2}\right)^2 - \sqrt{\pi}\right] \\ &= \left(x^2 - \frac{\pi}{2}\right)^2 - \left(\sqrt{\left(\frac{\pi}{2}\right)^2 - \sqrt{\pi}}\right)^2, \end{aligned}$$

provided $(\pi/2)^2 \geq \sqrt{\pi}$.

22. Solve the inequality

$$10^{-2}x^2 + (1 + 10^{-2})x + 1 > 0.$$

Solution.

$$\begin{aligned} 10^{-2}x^2 + (1 + 10^{-2})x + 1 > 0 &\iff x^2 + 101x + 100 > 0 \\ &\iff (x + 100)(x + 1) > 0 \\ &\iff x < -100 \text{ or } x > -1. \end{aligned}$$

23. Show that

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}.$$

Solution. By cross-multiplication it is enough to check

$$(-b \pm \sqrt{b^2 - 4ac})(-b \mp \sqrt{b^2 - 4ac}) = 4ac,$$

which is the difference of squares.

24. Consider $ax^2 + bx + c = 0$ with $a = 10^{-10}$, $b = 1 + 10^{-10}$, $c = 1$.

(a) Show that $b^2 - 4ac = (1 - 10^{-10})^2$.

Solution.

$$\begin{aligned} b^2 - 4ac &= (1 + 10^{-10})^2 - 4 \cdot 10^{-10} \\ &= 1 + 2 \cdot 10^{-10} + 10^{-20} - 4 \cdot 10^{-10} \\ &= 1 - 2 \cdot 10^{-10} + 10^{-20} = (1 - 10^{-10})^2. \end{aligned}$$

(b) Using both formulae of the previous item, show that the solutions are $x = -1$ and $x = -10^{10}$.

Solution.

$$x = \frac{-(1 + 10^{-10}) \pm (1 - 10^{-10})}{2 \cdot 10^{-10}}$$

gives $x = -1$ or $x = -10^{10}$. The reciprocal form

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

yields the same two roots.

25. Use

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

to solve $ax^2 + bx + c = 0$ when $a > 0$. Show all details.

Solution.

$$\begin{aligned} ax^2 + bx + c = 0 &\iff x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\ &\iff \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\ &\iff \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \end{aligned}$$

Hence

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

provided the discriminant is nonnegative.

26. Write $x^2 - 12x + 8$ as a difference of two squares.

Solution.

$$\begin{aligned}x^2 - 12x + 8 &= (x - 6)^2 - 36 + 8 \\&= (x - 6)^2 - 28 \\&= (x - 6)^2 - (\sqrt{28})^2.\end{aligned}$$

27. Write $x^2 - 10x + 7$ as a difference of two squares.

Solution.

$$\begin{aligned}x^2 - 10x + 7 &= (x - 5)^2 - 25 + 7 \\&= (x - 5)^2 - 18 \\&= (x - 5)^2 - (\sqrt{18})^2.\end{aligned}$$

28. Solve $10^{-2}x^2 + (1 + 10^{-2})x + 1 = 0$.

Solution.

$$\begin{aligned}10^{-2}x^2 + (1 + 10^{-2})x + 1 = 0 &\iff x^2 + 101x + 100 = 0 \\&\iff (x + 100)(x + 1) = 0 \\&\iff x = -100 \text{ or } x = -1.\end{aligned}$$