

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE QUESTIONS ON POLYNOMIALS

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____
Student #: _____ Professor: Philip Pennance Section: _____

1. Factor completely

$$3x^2 \cdot (3x + 4)^2 + x^3 \cdot 2 \cdot (3x + 4) \cdot 3.$$

Solution.

$$\begin{aligned} 3x^2 \cdot (3x + 4)^2 + x^3 \cdot 2 \cdot (3x + 4) \cdot 3 &= 3x^2 \cdot (3x + 4) [(3x + 4) + 2x] \\ &= 3x^2(3x + 4)(5x + 4). \end{aligned}$$

2. Find a polynomial $p(x)$ of degree 2 with real coefficients such that $p(7 + 12i) = 0$ and the coefficient of x^2 is 1.

Solution. Let $\alpha = 7 + 12i$. Since the coefficients are real, $\bar{\alpha} = 7 - 12i$ is also a root. Hence

$$\begin{aligned} p(x) &= (x - \alpha)(x - \bar{\alpha}) \\ &= x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} \\ &= x^2 - 14x + 193. \end{aligned}$$

3. Let $f(x) = x^4 + 14x^3 + 12x^2 + 87x + 5$.

(a) Use synthetic division to compute $f(-13)$.

(b) Find the polynomial $q(x)$ and the real number r such that

$$f(x) = q(x)(x + 13) + r.$$

Solution.

$$\begin{array}{r|rrrrr} -13 & 1 & 14 & 12 & 87 & 5 \\ & & -13 & -13 & 13 & -1300 \\ \hline & 1 & 1 & -1 & 100 & -1295 \end{array}$$

Thus $q(x) = x^3 + x^2 - x + 100$ and $r = -1295$, so $f(-13) = -1295$.

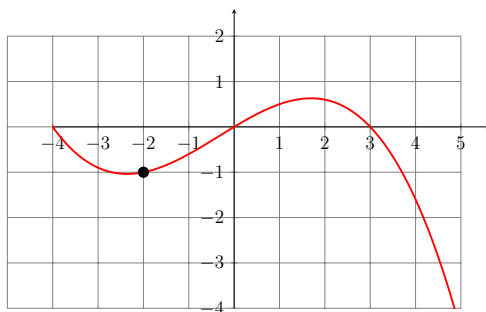
4. Find the three complex roots of $x^3 + 1 = 0$. **Solution.**

$$\begin{aligned} x^3 + 1 &= (x + 1)(x^2 - x + 1) \\ &= (x + 1) \left[\left(x - \frac{1}{2} \right)^2 + \frac{3}{4} \right]. \end{aligned}$$

Hence

$$x \in \left\{ -1, \frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right\}.$$

5. The polynomial p has degree 3. You may assume that all its roots are integers.



- (a) Find $p(-2)$, assuming it is an integer.

- (b) Find a formula for $p(x)$.

Solution. The marked point shows $p(-2) = -1$. The graph crosses the x -axis at the integers -4 , 0 and 3 , so

$$p(x) = cx(x + 4)(x - 3).$$

The condition $p(-2) = -1$ gives $20c = -1$, hence $c = -\frac{1}{20}$ and

$$p(x) = -\frac{1}{20} x(x + 4)(x - 3).$$

6. Find the two complex square roots of $w = 3 + 4i$.

Solution. If $z = a + ib$ satisfies $z^2 = w$, then

$$a^2 + b^2 = 5, \quad a^2 - b^2 = 3, \quad 2ab = 4.$$

The solutions are $z = 2 + i$ and $z = -2 - i$.

7. Let $z = 2 - i$ and $w = -3 + 2i$. Write each of the following in the form $a + bi$:

(a) $\frac{z}{w}$

(b) $(x - z)(x - \bar{z})$

Solution.

$$\frac{z}{w} = -\frac{8}{13} - \frac{1}{13}i, \quad (x - z)(x - \bar{z}) = x^2 - 4x + 5.$$

8. Write each of the following in the form $a + bi$:

(a) $i^{1234567654323456}$

(b) $|(3 + 4i)^8|$

Solution. $i^{1234567654323456} = 1$ and $|(3 + 4i)^8| = 5^8$.

9. Let $p(x) = x^3 - 4x^2 + 3x - 12$.

(a) Solve $p(x) = 0$ given that $i\sqrt{3}$ is a root.

(b) Write p as a product of irreducible factors over $\mathbb{C}$.

Solution. $p(x) = (x - 4)(x - i\sqrt{3})(x + i\sqrt{3})$, so the roots are 4, $\pm i\sqrt{3}$.

10. Let $z, w \in \mathbb{C}$. Prove that $\overline{zw} = \bar{z}\bar{w}$.

Solution. Write $z = a + bi$ and $w = u + vi$. Direct expansion of both sides yields

$$\overline{zw} = (au - bv) - i(av + bu) = \bar{z}\bar{w}.$$

11. Find a polynomial with integer coefficients of minimal degree that has -1 and i among its zeros.

Solution. The conjugate $\bar{i} = -i$ must also be a zero, so a monic polynomial of least degree is

$$(x + 1)(x^2 + 1) = x^3 + x^2 + x + 1.$$

12. If $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a root of $x^4 - 6x^2 - 7x - 6$, find all zeros.

Solution. The conjugate is also a root, giving the quadratic factor $x^2 + x + 1$. Division yields

$$x^2 - x - 6 = (x + 2)(x - 3).$$

The zeros are $\{-2, 3, \alpha, \bar{\alpha}\}$ where $\alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

13. Let $p \in \mathbb{R}[x]$. Show that if $\beta + i\gamma$ is a root of p , then $(x - \beta)^2 + \gamma^2$ is a factor of p .

Solution. The conjugate $\beta - i\gamma$ is also a root, so

$$(x - (\beta + i\gamma))(x - (\beta - i\gamma)) = (x - \beta)^2 + \gamma^2$$

divides p .

14. Find the quotient $q(x)$ and remainder r when $x^3 - 8$ is divided by $x - 3$.

Solution. $q(x) = x^2 + 3x + 9$ and $r = 19$.

15. Let $P(x) = x^3 - 7x^2 + 5x + 3$ and $D(x) = x - 1$.

(a) Use synthetic division to find the quotient and remainder when P is divided by D .

(b) Find numbers A, B, C, D such that

$$\frac{P(x)}{D(x)} = Ax^2 + Bx + C + \frac{D}{x - 1}.$$

Solution. Quotient $x^2 - 6x - 1$, remainder 2. Hence $A = 1$, $B = -6$, $C = -1$, $D = 2$.

16. Find b if one solution of $x^2 + bx + 1 = 0$ is twice the other.

Solution. $b = \pm \frac{3\sqrt{2}}{2}$.

17. If r_1 and r_2 are solutions of $x^2 + bx + c = 0$, prove that $r_1 r_2 = c$ and $r_1 + r_2 = -b$.

Solution.

$$x^2 + bx + c = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2.$$

Comparing coefficients gives the two identities.

18. Find a polynomial $q(x)$ and a constant r such that

$$x^4 - 16 = q(x)(x - 2) + r.$$

Solution. $q(x) = x^3 + 2x^2 + 4x + 8$ and $r = 0$.

19. Let $p(x) = x^2 + ax + b$. If $p(5) = -14$ and $x + 2$ is a factor of p , find $p(1)$.

Solution. $p(x) = x^2 - 5x - 14$, so $p(1) = -18$.

20. Find the quotient $Q(x)$ and remainder $R(x)$ when

$$2x^4 - 3x^3 + 10x^2 - x + 2$$

is divided by $x^2 - 5x + 1$. **Solution.**

$Q(x) = 2x^2 + 7x + 43$, $R(x) = 207x - 41$.

21. One solution of $x^3 - 8x^2 + 16x - 3 = 0$ is 3. Find the sum of the other two solutions.

Solution. The other factor is $x^2 - 5x + 1$, so the sum of its roots is 5.

22. Find k if $x - 2$ is a factor of

$$x^3 + (3 - k)x^2 + kx - 6.$$

Solution. By the factor theorem the value at $x = 2$ is zero:

$$8 + (3 - k) \cdot 4 + 2k - 6 = 0$$

$$14 - 2k = 0$$

$$k = 7.$$

23. Let

$$p(x) = x^5 + (3k - 1)x^4 - 5x^3 + (k + 2)x^2 + 1.$$

Find k if $x + 1$ is a factor of p .

Solution. $p(-1) = 0$ gives

$$-1 + (3k - 1) + 5 + (k + 2) + 1 = 0 \implies 4k + 6 = 0 \implies k = -\frac{3}{2}.$$

24. Let $w = -2 - 2i$. Write each of the following in the form $a + bi$:

(a) $\overline{w}$

(b) $|w|^2$

(c) $w\overline{w}$

(d) $\frac{1}{w}$

Solution.

$$\overline{w} = -2 + 2i, \quad |w|^2 = 8, \quad w\overline{w} = 8, \quad \frac{1}{w} = -\frac{1}{4} + \frac{1}{4}i.$$

25. Write $i^{1234567}$ in the form $a + bi$. **Solution.** $i^{1234567} = i^7 = i^3 = -i$.

26. Write $(1 + i)^4$ in the form $a + bi$. **Solution.** $(1 + i)^2 = 2i$, so $(1 + i)^4 = (2i)^2 = -4$.

27. Write $i^{33}(2 + i)$ in the form $a + bi$. **Solution.** $i^{33} = i$, hence $i(2 + i) = -1 + 2i$.

28. Write $\frac{75 + 125i}{3 - 4i}$ in the form $a + bi$. **Solution.** $-11 + 27i$.

29. Solve $w^2 = 3 - 4i$ over the complex numbers. **Solution.** $w = 2 - i$ and $w = -2 + i$.

30. Find i^{p+q} if p and q are the zeros of the quadratic $X^2 - 7X + C$.

Solution. By Vieta, $p + q = 7$, so $i^{p+q} = i^7 = -i$. (The value of C is not needed.)

31. Solve $x^2 + 25 = 2x - 75$ over the complex numbers.

Solution.

$$x^2 - 2x + 100 = 0 \implies x = 1 \pm i\sqrt{99} = 1 \pm 3i\sqrt{11}.$$

32. Let $p(x) = 1 - 2x - 3x^3$ and $q(x) = -4 - 5x - 6x^3$. Find

(a) $p(x)q(x)$

(b) $p(x) - q(x)$

Solution.

$$\begin{aligned} p(x)q(x) &= 18x^6 + 27x^4 + 6x^3 + 10x^2 + 3x - 4, \\ p(x) - q(x) &= 3x^3 + 3x + 5. \end{aligned}$$

33. Let

$$p(x) = x^4 + (h + 3k)x^2 + 3x + 2, \quad q(x) = x^4 + x^2 + (h - 3k)x + 2.$$

Find h and k if $p(x) = q(x)$.

Solution. Equating coefficients yields the system

$$h + 3k = 1, \quad h - 3k = 3.$$

Adding and subtracting gives $h = 2$ and $k = -\frac{1}{3}$.

34. Given $(-2x + 5)(3x - 4) = Ax^2 + Bx + C$, what is the value of $A + B$?

Solution. The product is $-6x^2 + 23x - 20$, so $A + B = 17$.

35. If $X^4 - 17$ is divided by $AX^2 + BX + C$, the quotient is $x^2 + 2$ and the remainder is -13 . Find A , B and C .

Solution.

$$X^4 - 17 = (AX^2 + BX + C)(x^2 + 2) - 13$$

is equivalent to $X^4 - 4 = (AX^2 + BX + C)(x^2 + 2)$. Comparing coefficients gives $A = 1$, $B = 0$, $C = -2$.

36. Prove that there do not exist integers a, b such that $5^{2/7} = a/b$.

Solution. $5^{2/7}$ is a root of $x^7 - 25$. By the rational-root theorem the only possible rational roots are $\pm 1, \pm 5, \pm 25$, none of which equals $5^{2/7}$.

37. Prove that there do not exist integers a, b such that $\sqrt[3]{17} = a/b$.

Solution. $\sqrt[3]{17}$ is a root of $x^3 - 17$. The only possible rational roots are $\pm 1, \pm 17$, none of which work.

38. Find the value of B if

$$(8x^2 + 4x + 7)(3x^2 + 5x + 6) = 24x^4 + 52x^3 + Bx^2 + 59x + 42.$$

Solution. Expanding the left-hand side produces $24x^4 + 52x^3 + 89x^2 + 59x + 42$, so $B = 89$.

39. If $x^4 + 81$ is divided by $ax^2 + bx + c$, the quotient is $x^2 - 3$ and the remainder is 90. Find a , b and c . **Solution.** $a = 1$, $b = 0$, $c = 3$.

40. Let $f(x) = x^5 - 169x^3 - 10x^2 - 129x + 12$. Find the polynomial $q(x)$ such that

$$f(x) = (x + 13)q(x) + f(-13).$$

Solution. $q(x) = x^4 - 13x^3 - 10x + 1$.

41. Solve $z^2 = 5 + 12i$.

Solution. If $z = a + ib$, then $a^2 + b^2 = 13$, $a^2 - b^2 = 5$ and $2ab = 12$. The solutions are $z = 3 + 2i$ and $z = -3 - 2i$.

42. Let $z = 3 + i$ and $w = -2 - 2i$. Find

(a) w^3

(b) $|w|^2$

(c) $z \cdot \bar{w}$

(d) $\frac{z}{w}$

Solution.

$$w^3 = 16 - 16i, \quad |w|^2 = 8, \quad z\bar{w} = -8 + 4i, \quad \frac{z}{w} = -1 + \frac{1}{2}i.$$

43. Find a and b if $x - 1$ is a factor of both

$$p(x) = x^3 + x^2 + ax + b \quad \text{and} \quad q(x) = x^3 - x^2 - ax + b.$$

Solution. $p(1) = 0$ and $q(1) = 0$ give $a + b = -2$ and $-a + b = 0$. Hence $a = b = -1$.

44. Identify the mistake (if any) in the following synthetic division of $x^4 + 3x^2 - 4x - 2$ by $x - 2$:

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & -4 & -2 & \\ & & 2 & 10 & 12 & \\ \hline & 1 & 5 & 6 & 10 & \end{array}$$

Solution. The dividend is degree 4, so the missing x^3 coefficient 0 must be included. The correct tableau is

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 3 & -4 & -2 & \\ & & 2 & 4 & 14 & 20 & \\ \hline & 1 & 2 & 7 & 10 & 18 & \end{array}$$

45. Find the remainder when

$$x^{1000} + x^{100} - 3x^{12} + 1$$

is divided by $x - i$. **Solution.** By the remainder theorem the remainder is $p(i) = 0$.

46. Show that $a + b$ is a factor of

$$abc - (a + b + c)(bc + ca + ab)$$

and find the other factor. **Solution.** The expression equals $(a + b)(ab + bc + ca - c^2)$.

47. Solve $x^2 = -17$ over the complex numbers. **Solution.** $x = \pm i\sqrt{17}$.