

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

QUESTIONS ON FRACTIONS, THE NUMBER LINE AND THE DIVISION THEOREM

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____
Student #: _____ Professor: Philip Pennance Section: _____

1. Prove the following statements using ONLY the **definition of fraction** (rational number) in the notes. Do NOT use computations. Do NOT use the concept of division.
 - (a) 2 is a fraction.

Solution. Definition (from the notes). Let k and ℓ be whole numbers. The *fraction* $\frac{k}{\ell}$ is the point on the number line obtained by subdividing each of the unit intervals $[0, 1]$, $[1, 2]$, $\dots$ into ℓ segments of equal length and counting k markers to the right, starting at the origin.

The point 2 is obtained by subdividing each unit interval into one piece (that is, by doing nothing) and counting two markers to the right from the origin. Hence 2 is the fraction $\frac{2}{1}$. Remark.

Many students said that a fraction is “part of a whole.” That is not a definition: it has no genus, and “part” and “whole” are not mathematical terms.

(b) $\frac{99}{100} > \frac{98}{99}$

Solution. If each unit interval is divided into 100 equal parts, then $\frac{99}{100}$ is the mark immediately preceding 1, so its distance from 1 is $\frac{1}{100}$. If each unit interval is divided into 99 equal parts, then $\frac{98}{99}$ is the mark immediately preceding 1, so its distance from 1 is $\frac{1}{99}$. Since $\frac{1}{100} < \frac{1}{99}$, the point $\frac{99}{100}$ is closer to 1. Both points lie to the left of 1, so

$$\frac{99}{100} > \frac{98}{99}.$$

2. Find natural numbers n and m such that

$$\frac{n}{m} = 12.11\overline{234}.$$

(Do not simplify.)

Solution.

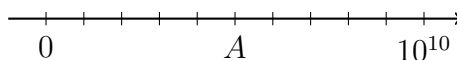
$$\begin{aligned}x &= 12.11\overline{234}, \\10^2x &= 1211.\overline{234}, \\10^5x &= 1211234.\overline{234}.\end{aligned}$$

Subtracting yields

$$x = \frac{1211234 - 1211}{10^5 - 10^2}.$$

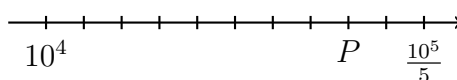
Thus one may take $n = 1211234 - 1211$ and $m = 10^5 - 10^2$.

3. Find the point A .



Solution. There are ten subdivisions between 0 and 10^{10} , so each has length 10^9 . The point A is five subdivisions from the origin, hence $A = 5 \times 10^9$.

4. Prove that the point P is a fraction (a rational number).



Solution. Let Δ be the length of each subinterval. There are ten subintervals between 10^4 and $\frac{10^5}{5}$, so

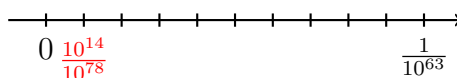
$$\Delta = \frac{\frac{10^5}{5} - 10^4}{10}.$$

Thus Δ is a fraction, and so is 8Δ . A sum of fractions is a fraction, hence

$$P = 10^4 + 8\Delta$$

is a fraction.

5. Locate the fraction $\frac{10^{14}}{10^{78}}$ on the following scale.

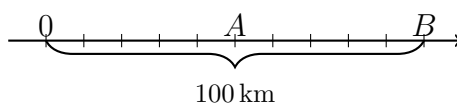


Solution.

$$\frac{10^{14}}{10^{78}} = \frac{1}{10^{64}} = \frac{1}{10} \cdot \frac{1}{10^{63}}.$$

So the point lies one tenth of the way from 0 to $\frac{1}{10^{63}}$.

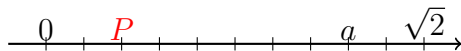
6. Find the distance between the points A and B in millimeters.



Solution.

$$d(A, B) = 50 \text{ km} = 5 \times 10^7 \text{ mm}.$$

7. Locate the point $P = \sqrt{2} - a$ on the following scale.

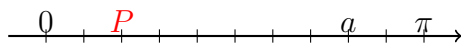


Solution.

$$d(a, \sqrt{2}) = |\sqrt{2} - a| = d(\sqrt{2} - a, 0).$$

Hence $P = \sqrt{2} - a$ is as far from the origin as a is from $\sqrt{2}$. On the given scale that distance is two tenths of the segment, so P lies at the second tick.

8. Locate the point $P = \pi - a$ on the following scale.

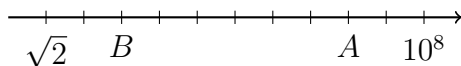


Solution.

$$d(a, \pi) = |\pi - a| = d(\pi - a, 0).$$

Thus P is as far from the origin as a is from π .

9. Let A and B be the points on the following scale:



If $P = \frac{2}{3}B + \frac{1}{3}A$:

- (a) Show that $|P - A| = \frac{2}{3}|B - A|$.

Solution.

$$|P - A| = \left| \left(\frac{2}{3}B + \frac{1}{3}A \right) - A \right| = \left| \frac{2}{3}B - \frac{2}{3}A \right| = \frac{2}{3}|B - A|.$$

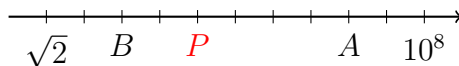
Thus $d(P, A)$ is two thirds of $d(A, B)$.

Remark. It is easy to check that $B < P < A$.

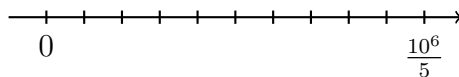
- (b) Find the ratio $\frac{d(P, A)}{d(P, B)}$.

Solution. From (a), $d(P, A) = \frac{2}{3}d(A, B)$. A similar computation gives $d(P, B) = \frac{1}{3}d(A, B)$. Hence the ratio is 2.

- (c) Locate P on the scale.



10. (a) Find the distance Δ between successive marks on the following scale.



Solution.

$$\Delta = \frac{\frac{10^6}{5} - 0}{10} = \frac{10^5}{5}.$$

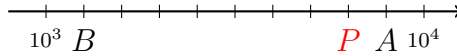
- (b) Locate the fraction $\frac{10^3}{5}$ on the scale above.

Solution. Each subdivision has length $\Delta = \frac{10^5}{5}$.

$$\frac{10^3/5}{\Delta} = \frac{10^3}{10^5} = \frac{1}{100}.$$

Thus $\frac{10^3}{5}$ lies one hundredth of a division to the right of the origin, and on the given scale it is almost indistinguishable from the origin.

11. Locate the point $P = \frac{1}{8}B + \frac{7}{8}A$ on the following scale.

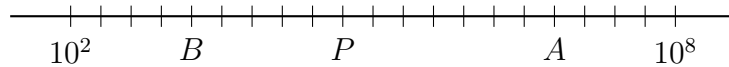


Solution. Let Δ be the distance between successive marks. Then $B = 10^3 + \Delta$ and $A = 10^3 + 9\Delta$, so

$$P = \frac{1}{8}(10^3 + \Delta) + \frac{7}{8}(10^3 + 9\Delta) = 10^3 + 8\Delta.$$

Hence P lies eight divisions to the right of 10^3 .

12. Let A and B be the points on the following scale.



If $P = \frac{7}{12}B + \frac{5}{12}A$:

- (a) Find the ratio $\frac{d(P, A)}{d(P, B)}$.

Solution.

$$d(P, A) = |P - A| = \left| \frac{7}{12}B - \frac{7}{12}A \right| = \frac{7}{12} d(A, B).$$

A similar computation gives $d(P, B) = \frac{5}{12} d(A, B)$. Hence

$$\frac{d(P, A)}{d(P, B)} = \frac{7}{5}.$$

- (b) Prove that B is a natural number.

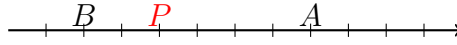
Solution. There are 20 equal divisions between 10^2 and 10^8 , so

$$\Delta = \frac{10^8 - 10^2}{20} = 5(10^6 - 1).$$

Thus Δ is a natural number, and so is

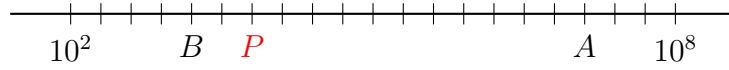
$$B = 10^2 + 4\Delta.$$

13. Without knowing the origin, locate the point $\frac{2}{3}B + \frac{1}{3}A$ on the following scale.



Solution. $P = \frac{2}{3}B + \frac{1}{3}A$ divides the segment from B to A in the ratio $1 : 2$, so P lies one third of the way from B to A .

14. Let A and B be the points on the following scale.



- (a) Prove that B is a natural number.

Solution. As before,

$$\Delta = \frac{10^8 - 10^2}{20} = 5(10^6 - 1),$$

which is a natural number. The mark B is four divisions from 10^2 , so

$$B = 10^2 + 4\Delta$$

is a natural number.

- (b) Find the exact value of

$$P = \frac{2}{13}A + \frac{11}{13}B$$

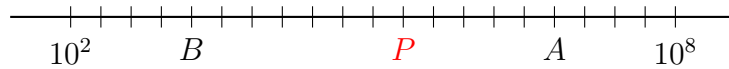
and locate it on the scale.

Solution. Here $A = 10^2 + 17\Delta$ and $B = 10^2 + 4\Delta$, so

$$P = 10^2 + 6\Delta.$$

That is the sixth tick to the right of 10^2 , as marked above.

15. Let A and B be the points on the following scale.



- (a) Prove that B is a natural number.

Solution. $\Delta = 5(10^6 - 1)$ is a natural number and $B = 10^2 + 4\Delta$.

- (b) Locate $P = \frac{7}{12}A + \frac{5}{12}B$ on the scale.

Solution. With $A = 10^2 + 16\Delta$ and $B = 10^2 + 4\Delta$,

$$P = 10^2 + 11\Delta,$$

which is the eleventh tick, as marked.

Additional remarks. Let λ_1, λ_2 be positive real numbers with $\lambda_1 + \lambda_2 = 1$. The combination $P = \lambda_1 A + \lambda_2 B$ does not depend on the choice of origin, and

$$d(P, A) : d(P, B) = \lambda_2 : \lambda_1$$

(note the order). The same invariance appears in the midpoint formula $P = \frac{1}{2}A + \frac{1}{2}B$ and in the center-of-mass formula: if mass m_1 is at A and mass m_2 is at B , then

$$X_{\text{CM}} = \frac{m_1}{m_1 + m_2}A + \frac{m_2}{m_1 + m_2}B.$$

The coefficients again sum to 1. The same pattern appears in a grade-point average.

16. Find natural numbers q and r such that

$$1234567 = 16q + r, \quad 0 \leq r < 16.$$

Show all work.

Solution (division algorithm).

$$\begin{array}{rcl} 1 & = & \boxed{0} \times 16 + 1 \\ 12 & = & \boxed{0} \times 16 + 12 \\ 123 & = & \boxed{7} \times 16 + 11 \\ 114 & = & \boxed{7} \times 16 + 2 \\ 25 & = & \boxed{1} \times 16 + 9 \\ 96 & = & \boxed{6} \times 16 + 0 \\ 7 & = & \boxed{0} \times 16 + 7 \end{array}$$

Hence

$$1234567 = 16 \times 77160 + 7.$$

Remark. At each step the remainder is multiplied by 10 and the next digit is added. In the usual layout this is disguised as “bringing down” a digit. “Bringing down” is not a mathematical notion; “down” depends on gravity or on how the page is turned.

17. Let $n = 1234$ and $m = 16$. Find $q, r \in \mathbb{N}$ such that

$$n = q \cdot m + r, \quad 0 \leq r < m.$$

Solution.

$$\begin{array}{rcl} 1 & = & \boxed{0} \times 16 + 1 \\ 12 & = & \boxed{0} \times 16 + 12 \\ 123 & = & \boxed{7} \times 16 + 11 \\ 114 & = & \boxed{7} \times 16 + 2 \end{array}$$

Hence

$$1234 = 16 \times 77 + 2.$$

18. Let $n = 12346$ and $m = 16$. Find $q, r \in \mathbb{N}$ such that

$$n = q \cdot m + r, \quad 0 \leq r < m.$$

Solution.

$$\begin{aligned} 1 &= \boxed{0} \times 16 + 1 \\ 12 &= \boxed{0} \times 16 + 12 \\ 123 &= \boxed{7} \times 16 + 11 \\ 114 &= \boxed{7} \times 16 + 2 \\ 26 &= \boxed{1} \times 16 + 10 \end{aligned}$$

Hence

$$12346 = 16 \times 771 + 10.$$

19. If $A < B$ and $P = \frac{2}{3}B + \frac{1}{3}A$, show that $A < P < B$.

Solution. Since $A < B$, one has $B - A > 0$. Then

$$P - A = \frac{2}{3}(B - A) > 0,$$

so $A < P$. Likewise

$$B - P = \frac{1}{3}(B - A) > 0,$$

so $P < B$. Combining these yields $A < P < B$.

20. Find (in decimal form) an irrational number between 2 and $\frac{21}{20}$.

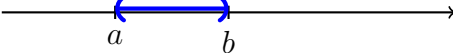
Solution. Note that $\frac{21}{20} = 1.05 < 2$. The number $\sqrt{2} = 1.414213\dots$ satisfies

$$\frac{21}{20} < \sqrt{2} < 2$$

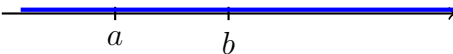
and is irrational.

21. Let a and b be real numbers with $a < b$. Draw the following sets on the line.

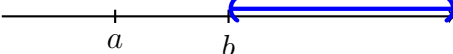
(a) $a < x < b$



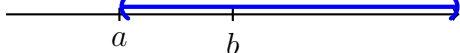
(b) $a < x \vee x < b$



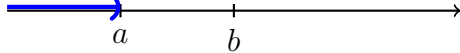
(c) $a < x \wedge x > b$



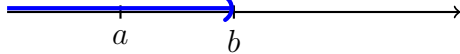
(d) $a < x \vee x > b$



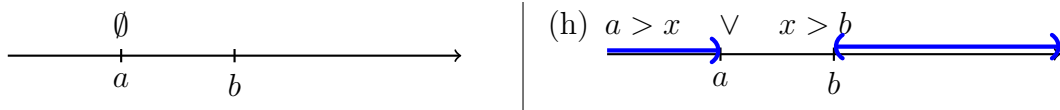
(e) $a > x \wedge x < b$



(f) $a > x \vee x < b$



(g) $a > x \wedge x > b$



22. Use the following information to find integers q and r such that

$$123456 = 16q + r, \quad 0 \leq r < 16.$$

$$\begin{aligned} 1 &= 0 \times 16 + 1 \\ 12 &= 0 \times 16 + 12 \\ 123 &= 7 \times 16 + 11 \\ 114 &= 7 \times 16 + 2 \\ 25 &= 1 \times 16 + 9 \\ 96 &= 6 \times 16 + 0 \end{aligned}$$

$$q = \underline{\hspace{2cm}} \quad r = \underline{\hspace{2cm}}$$

Solution. The quotient digits are 0, 0, 7, 7, 1, 6, so $q = 7716$ and $r = 0$:

$$123456 = 16 \times 7716 + 0.$$

23. Find natural numbers n and m such that

$$\frac{n}{m} = 12.11\overline{234}.$$

(Do not simplify.)

Solution. See item 3. One may take $n = 1211234 - 1211$ and $m = 10^5 - 10^2$.