

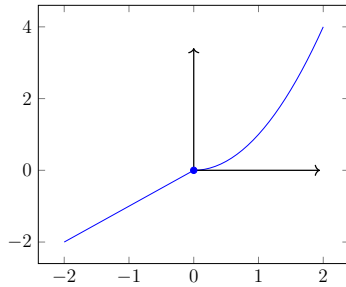
Graphing – Examples

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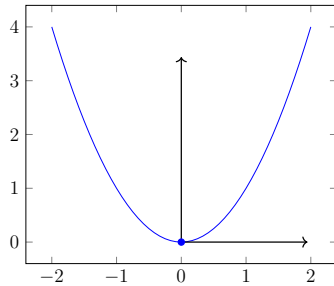
- Two functions f and g are said to be *equal* (written $f = g$) if their domains, codomains and graphs are all equal. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

- Sketch the graph of the function g given by $g(x) = f(|x|)$.
- Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(x) = x^2$, $x \in \mathbb{R}$. Determine, using the definition of equality whether or not $h = g$. Justify your answer.

Solution.



Graph of f

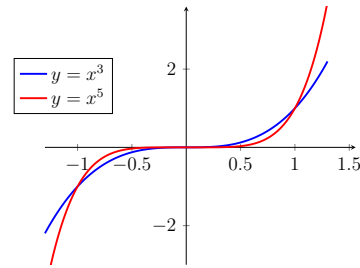


Graph of g

The functions g and h are equal since they have the same domain $\mathbb{R}$, the same codomain $\mathbb{R}$ and the same graph $\{(x, x^2) : x \in \mathbb{R}\}$.

- Show that $0 < x < 1 \implies x^3 > x^5$.
 - Sketch, in the same coordinate system, the graphs of the following functions.

Solution.



- $f(x) = x^5$, $x \in \mathbb{R}$.
- $f(x) = x^3$ $x \in \mathbb{R}$.

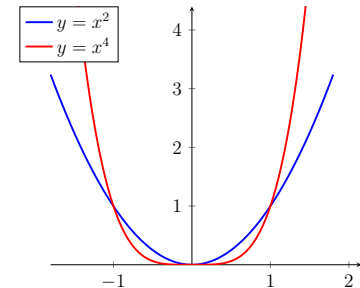
Remark: Sketches of graphs are very much scale dependent and it may not be possible to draw two graphs on the same scale.

- Sketch, in the same coordinate system, the graphs of the following functions.

- $f(x) = x^2$, $x \in \mathbb{R}$.
- $f(x) = x^4$ $x \in \mathbb{R}$.

Remark: $-1 < x < 1 \implies x^2 > x^4$.

Solution.



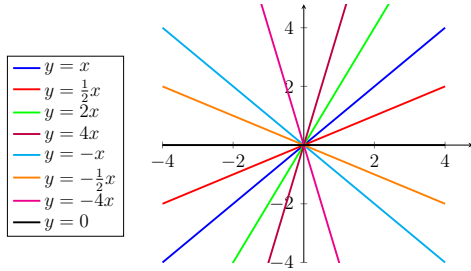
- Sketch, in the same coordinate system, the graphs of the following sets. Do not forget to label the graphs. If possible use different colors for different graphs.

- $\{(x, y) : y = 0\}$
- $\{(x, y) : y = \frac{x}{2}\}$
- $\{(x, y) : y = x\}$
- $\{(x, y) : y = 2x\}$

¹<https://pennance.us>

- (e) $\{(x, y) : y = 4x\}$
- (f) $\{(x, y) : x = 0\}$
- (g) $\{(x, y) : y = -\frac{x}{2}\}$
- (h) $\{(x, y) : y = -x\}$
- (i) $\{(x, y) : y = -2x\}$
- (j) $\{(x, y) : y = -4x\}$

Solution.



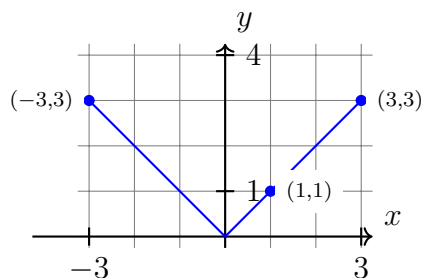
Remarks:

- (a) Notice that $y = -mx$ is a reflection of $y = mx$ in the horizontal axis.
 - (b) The vertical axis $\{(x, y) : x = 0\}$ is not the graph of a function.
5. Let x be a real number. Let $d(x, 0)$ denote the distance between x and the origin. Since each real number has a unique distance from the origin, $f = d(x, 0)$ is a function. Draw a graph of the f in the Cartesian plane.

Solution.

Since distance is non-negative:

- (a) If $x \geq 0$ then $d(x, 0) = x$
- (b) If $x < 0$ then $-x > 0$ and so $d(x, 0) = -x$.



6. Let $x \in \mathbb{R}$ and define $h(x)$ be the maximum of the set $\{x, -x\}$. Show that h is not an injective function. Draw the graph of h .

Solution.

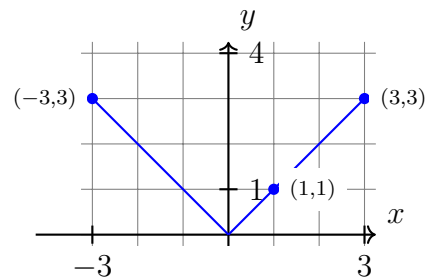
- (a) If $x > 0$ then $-x < 0$ and so

$$\max\{x, -x\} = x$$

- (b) If $x < 0$ then $-x > 0$ and so

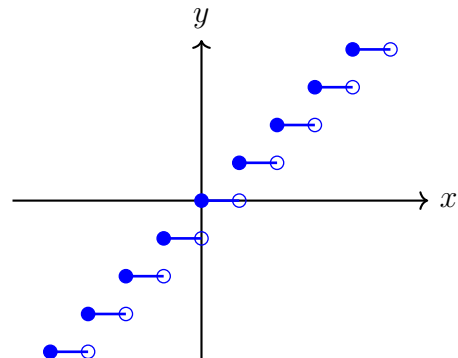
$$\max\{x, -x\} = -x.$$

Since $h(3) = h(-3)$ the function is not injective.

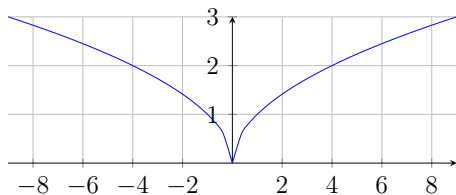


7. Let x be a real number and let $f : \mathbb{R} \rightarrow \mathbb{Z}$ be the function such that $f(x)$ is maximum integer which is less than or equal to x . Draw a graph of f in the Cartesian plane.

Solution.



8. Sea $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0. \end{cases}$ Dibuje la gráfica de la función $h(x) = f(|x|)$.



9. Let a be a real number and define a^+ by:

$$a^+ = \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

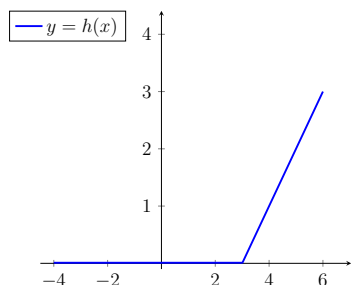
Draw the graphs of the following functions:

(a) $h(x) = (x - 3)^+$.

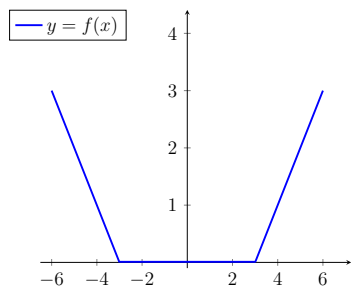
(b) $f(x) = (|x| - 3)^+$.

Solution.

$$h(x) = \begin{cases} x - 3 & \text{if } x \geq 3, \\ 0 & \text{if } x < 3 \end{cases}$$



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10. Indique el dominio natural sobre los reales: $f(x) = \frac{1}{\sqrt{x}} + \frac{x}{x+1}$

Solution.

Since the domain of a sum is the intersection of the domains,

$$D_f = \{x : x > 0\}.$$

11. Find the largest possible domain (over the real numbers) for the following functions.

(a) $f(x) = \frac{2}{\sqrt{9-x^2}}$

Solution.

$$-3 < x < 3.$$

(b) $f(x) = \frac{1}{1 + \frac{1}{x+1}}$

Solution.

$$\mathbb{R} \setminus \{-1, -2\}.$$

(If $x = -2$ then $\frac{1}{x+1} = -1$).

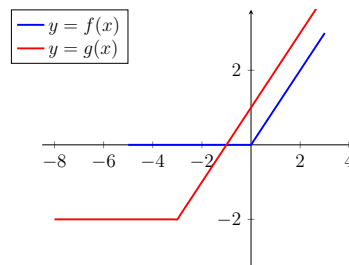
12. Let $f(x) = \frac{x+|x|}{2}$, $x \in \mathbb{R}$. A function g is defined by $g(x) = f(x+3) - 2$.

(a) Draw the graph of f ,

(b) Draw the graph of g

Solution.

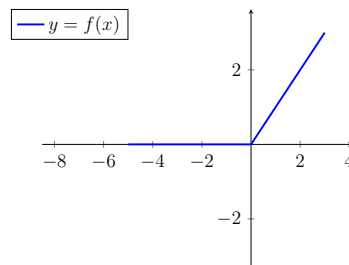
The graph of g is a translation of the graph of f by the vector $(-3, -2)$.



13. Let $f(x) = \frac{x+|x|}{2}$, $x \in \mathbb{R}$. A function g is defined by $g(x) = f(|x|)$.

(a) Draw the graph of f .

(b) Show that $g(x) = |x|$.



14. Two functions f and g are said to be *equal* (written $f = g$) if their domains, codomains and graphs are all equal. Let functions $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ be given, respectively by

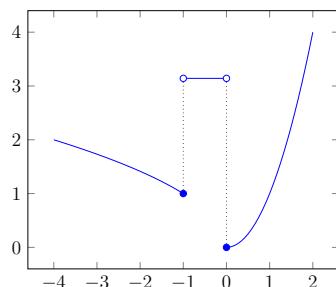
$$f(x) = \begin{cases} |x|^2 & \text{if } x \geq 0, \\ \pi & \text{if } -1 < x < 0 \\ \sqrt{-x} & \text{if } x \leq -1. \end{cases}$$

$$g(x) = x^2$$

$$h(x) = f(|x|)$$

- (a) Sketch the graph of f .
 (b) Determine, using the definition of equality of functions, whether or not $h = g$. Justify your answer.

Solution.



The functions g and h are equal since they have the same domain $\mathbb{R}$, the same codomain $\mathbb{R}$ and the same graph $\{(x, x^2) : x \in \mathbb{R}\}$.

15. Let $f[a, b] \rightarrow \mathbb{R}$ be an arbitrary function. A function g is defined by $g(x) = f(x+3) - 2$. Show that

$$(x, y) \in G_f \iff (x-3, y-2) \in G_g.$$

What is the geometric significance of this?

Solution.

$$(x, y) \in G_f$$

$$\iff y = f(x)$$

$$\iff y = f((x-3)+3)$$

$$\iff y-2 = f((x-3)+3) - 2$$

$$\iff y-2 = g(x-3)$$

$$\iff (x-3, y-2) \in G_g.$$

Hence the graph of G is a translation of the graph of f by the vector $(-3, -2)$, i.e., by 3 units left and 2 units down.

16. Example.

Let f be a real valued function with domain $D_f \subseteq \mathbb{R}$. Let $g(x) = f(x+c)$ where $c > 0$. Show that the graph G_g of g is a translation to the left of G_f by c units.

Solution.

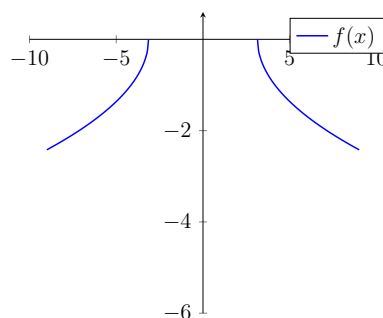
$$\begin{aligned} (x, y) \in G_f &\iff y = f(x) \\ &\iff y = f((x-c)+c) \\ &\iff y = g(x-c) \\ &\iff (x-c, y) \in G_g. \end{aligned}$$

Hence for any point (x, y)

$$(x, y) \in G_f \iff (x-c, y) \in G_g$$

which means that the graph of g is a translation to the left of G_f by c units.

17. Dibuje la gráfica de $y = -\sqrt{|x| - \pi}$.



18. Dibuje la gráfica y halle el campo de valores (imagen) de la función

$$f(x) = \max\{x, 1-x\}, \quad x \in \mathbb{R}.$$