

UNIVERSITY OF PUERTO RICO - RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE QUESTIONS ON FUNCTIONS

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____
Student # : _____ Professor: Philip Pennance Section: _____

1. Identify the only true statement.
 - (a) f is a function if f is a relation.
 - (b) f is a function if for every element x of the domain there exists a unique element y in the codomain.
 - (c) For f to be a relation it is necessary that f be a function.
 - (d) For f to be a relation it is sufficient that f be a function.

2. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *linear* if

- (a) $f(x+y) = f(x) + f(y)$,
- (b) $f(cx) = cf(x)$,

for all $x, y, c \in \mathbb{R}$.

Identify the two false statements.

- (a) The Celsius to Fahrenheit conversion function $F(C) = (9/5)C + 32$ is linear.
- (b) The function that converts feet to inches is linear.
- (c) $f(t) = 4t$ is linear.
- (d) If f is linear, then $f(5) = f(1) + f(1) + f(3)$ is true.
- (e) If $b \neq 0$, then $f(x) = mx + b$ is linear.

3. A student attempts to define a function f , with domain the set of non-negative rational numbers and codomain $\mathbb{N}$, by the equation

$$f(m/n) = m, \quad m, n \in \mathbb{N} \text{ and } n \neq 0.$$

Is f a function? Explain.

Solution. f is not a function. Suppose, to the contrary, that f is a function. Let G_f be its graph. Then

$$\left(\frac{1}{2}, 1\right) \in G_f \quad \text{and} \quad \left(\frac{8}{16}, 8\right) \in G_f.$$

But $\frac{1}{2} = \frac{8}{16}$, so in the corresponding bipartite graph the element $\frac{1}{2}$ has degree greater than 1, which contradicts the definition of a function.

4. Find the natural domain of the function

$$f(x) = \frac{1}{\sqrt{x}} + \frac{x}{x+1}.$$

Solution. Since the domain of a sum is the intersection of the domains,

$$D_f = \{x : x > 0\}.$$

5. Find the largest possible domain (over the real numbers) for each of the following functions.

(a) $f(x) = \frac{2}{\sqrt{9-x^2}}$

(b) $f(x) = \frac{1}{1 + \frac{1}{x+1}}$

Solutions.

- (a) $-3 < x < 3$.
- (b) $\mathbb{R} \setminus \{-1, -2\}$.

6. Find the domain of the function

$$f(x) = \frac{\sqrt{x+1}}{x}.$$

7. Let $f(x) = x + 3$ and $g(x) = x - 3$. Find:

- (a) $(fg)(3)$ (c) $(f + g)(3)$
 (b) $(f \circ g)(3)$ (d) $\left(\frac{f}{g}\right)(3)$

Solution.

$$(fg)(3) = f(3)g(3) = 6 \cdot 0 = 0.$$

$$(f \circ g)(3) = f(g(3)) = f(0) = 3.$$

$$(f + g)(3) = f(3) + g(3) = 6 + 0 = 6.$$

$(f/g)(3)$ does not exist, since $g(3) = 0$.

8. Let $a \in \mathbb{R}$ and $f: (-a, a) \rightarrow \mathbb{R}$. Define functions f_e and f_o by

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

and

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)].$$

- (a) Show that f_e is even.
 (b) Show that f_o is odd.
 (c) Write $f(x) = (x - 1)^2$ as the sum of an even function and an odd function.

Solution.

$$\begin{aligned} f_e(-x) &= \frac{1}{2}[f(-x) + f(-(-x))] \\ &= \frac{1}{2}[f(x) + f(-x)] \\ &= f_e(x). \end{aligned}$$

$$\begin{aligned} f_o(-x) &= \frac{1}{2}[f(-x) - f(-(-x))] \\ &= -\frac{1}{2}[f(x) - f(-x)] \\ &= -f_o(x). \end{aligned}$$

Let $f(x) = (x - 1)^2$, then $f_o(x) = -2x$ and $f_e(x) = x^2 + 1$ and so

$$(x - 1)^2 = (x^2 + 1) + (-2x).$$

9. Let $f: [a, b] \rightarrow \mathbb{R}$ be an arbitrary function. A function g is defined by

$$g(x) = f(x + 3) - 2.$$

Show that

$$(x, y) \in G_f \iff (x - 3, y - 2) \in G_g.$$

What is the geometric significance of this?

Solution.

$$\begin{aligned} (x, y) \in G_f &\iff y = f(x) \\ &\iff y = f[(x - 3) + 3] \\ &\iff y - 2 = f[(x - 3) + 3] - 2 \\ &\iff y - 2 = g(x - 3) \\ &\iff (x - 3, y - 2) \in G_g. \end{aligned}$$

Hence the graph of g is a translation of the graph of f by the vector $(-3, -2)$, that is, 3 units to the left and 2 units down.

10. If $f(x) = x^2$ and $g(x) = \sqrt{9 - x}$, find:

- (a) the composition $(g \circ f)(x)$;
 (b) show that $g \circ f$ is even.

11. Let $f(x) = x$, $g(x) = x + 1$, and $h(x) = \frac{1}{x}$. Find the compositions:

- (a) $(g \circ f)(x)$ (c) $(h \circ g)(x)$
 (b) $(f \circ h)(x)$ (d) $(h \circ h \circ f)(x)$

Solution.

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x) \\ &= x + 1. \end{aligned}$$

$$\begin{aligned} f \circ h(x) &= f(h(x)) \\ &= h(x) \\ &= x + 1. \end{aligned}$$

$$\begin{aligned} h \circ g(x) &= h(g(x)) \\ &= \frac{1}{x + 1}. \end{aligned}$$

$$\begin{aligned}
h \circ h \circ f(x) &= h(h(f(x))) \\
&= h(h(x)) \\
&= h(1/x) \\
&= x, \quad x \neq 0.
\end{aligned}$$

12. Let f be a real-valued function with domain $D_f \subseteq \mathbb{R}$. Let $g(x) = f(x + c)$ where $c > 0$. Show that the graph G_g of g is a translation to the left of G_f by c units.

Solution.

$$\begin{aligned}
(x, y) \in G_f &\iff y = f(x) \\
&\iff y = f((x - c) + c) \\
&\iff y = g(x - c) \\
&\iff (x - c, y) \in G_g.
\end{aligned}$$

Hence, for any point (x, y) ,

$$(x, y) \in G_f \iff (x - c, y) \in G_g,$$

which means that the graph of g is a translation to the left of G_f by c units.

13. Find the image of

$$f(x) = \frac{2x - 3}{x + 3}$$

and sketch the graph.

14. If $g(x) = 4x - 3$, $h(x) = 3x - 1$, and $k(x) = 5x - 7$, find

$$[(h \circ g) \circ k](-2).$$

15. Does there exist a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x^2) = x$? Justify your answer.

16. Find α and β if the set

$$G = \{(1, 4), (3, \alpha), (1, \beta), (3, 1)\}$$

is the graph of an injective function.

17. If $f(x) = x^2$ and $g(x) = \frac{1}{\sqrt{9 - x}}$, find:

- (a) the composition $(g \circ f)(x)$;
(b) the domain and range of $g \circ f$.

18. Determine whether the following functions are even, odd, neither, or both. Sketch their graphs.

(a) $f(x) = x^2, \quad x \geq 0$

(b) $f(x) = 0, \quad x \in \mathbb{R}$

(c) $f(x) = 0, \quad -1 < x < 2$

(d) $f(x) = \frac{x^3}{x^2 + x^4}, \quad x \in (-2, 0) \cup (0, 2)$

19. Let $f(x) = x^2 - 3x$. Show that

$$\frac{f(x + h) - f(x)}{h} = 2x + h - 3.$$

Solution.

$$\begin{aligned}
&\frac{f(x + h) - f(x)}{h} \\
&= \frac{((x + h)^2 - 3(x + h)) - (x^2 - 3x)}{h} \\
&= \frac{(x^2 + 2hx + h^2 - 3x - 3h) - (x^2 - 3x)}{h} \\
&= \frac{2hx + h^2 - 3h}{h} \\
&= 2x + h - 3.
\end{aligned}$$

20. Let $f(x) = 2x^2 - 3x - 4$. Show that

$$\frac{f(x + h) - f(x)}{h} = 4x + 2h - 3.$$

Solution.

$$\begin{aligned}
&\frac{f(x + h) - f(x)}{h} \\
&= \frac{(2(x + h)^2 - 3(x + h) - 4) - (2x^2 - 3x - 4)}{h} \\
&= \frac{4hx + 2h^2 - 3h}{h} \\
&= 4x + 2h - 3.
\end{aligned}$$