

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE QUESTIONS ON INDUCTION

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____

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1. Prove by induction that

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}.$$

Solution. Let $p(n)$ denote the predicate

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}.$$

The induction basis $p(1)$ is the statement

$$3 = \frac{3 \cdot 1 \cdot (1+1)}{2},$$

which is true. Let $n \in \mathbb{N}$ with $n \geq 1$, and assume (for this n) that

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}.$$

Then

$$\begin{aligned} 3 + 6 + 9 + \cdots + 3n + 3(n+1) &= \frac{3n(n+1)}{2} + 3(n+1) \\ &= \frac{3n(n+1)}{2} + \frac{6(n+1)}{2} \\ &= \frac{3(n+1)(n+2)}{2}. \end{aligned}$$

Hence if $p(n)$ is true, then so is $p(n+1)$. By mathematical induction, $p(n)$ holds for every $n \in \mathbb{N}$ with $n \geq 1$.

2. Let $x \in \mathbb{R}$ with $x \neq 1$. Prove by induction that

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

Solution. Let $p(n)$ denote the predicate

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

The statement $p(1)$ is

$$1 = \frac{1 - x^1}{1 - x},$$

which is true. Let $n \in \mathbb{N}$ with $n \geq 1$, and assume (for this n) that

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

Then

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{n-1} + x^n &= \frac{1 - x^n}{1 - x} + x^n \\ &= \frac{1 - x^n}{1 - x} + \frac{x^n(1 - x)}{1 - x} \\ &= \frac{1 - x^{n+1}}{1 - x}. \end{aligned}$$

Hence if $p(n)$ is true, then so is $p(n + 1)$. By mathematical induction, $p(n)$ holds for every $n \in \mathbb{N}$ with $n \geq 1$.

3. (a) Prove by induction that

$$(1.001)^n \geq 1 + n(0.001)$$

for every $n \in \mathbb{N}$.

Solution. Let $p(n)$ be the predicate $(1.001)^n \geq 1 + n(0.001)$. Then $p(1)$ is the statement

$$(1.001)^1 \geq 1 + 1(0.001),$$

which is true. Let $n \in \mathbb{N}$ and suppose that $p(n)$ holds, that is,

$$(1.001)^n \geq 1 + n(0.001).$$

Then

$$\begin{aligned} (1.001)^{n+1} &= (1.001)(1.001)^n \\ &\geq (1.001)(1 + n(0.001)) \\ &= 1 + (n + 1)(0.001) + n(0.001)^2 \\ &\geq 1 + (n + 1)(0.001). \end{aligned}$$

Hence $p(n + 1)$ is true whenever $p(n)$ is true. By induction, $p(n)$ holds for every $n \geq 1$.

- (b) Prove that $(1.001)^n \geq 10^3$ for every $n \geq 10^6$.

Solution. If $n \geq 10^6$, then part (a) yields

$$\begin{aligned} (1.001)^n &= (1 + 0.001)^n \\ &\geq 1 + n(0.001) \\ &\geq 1 + 10^6(0.001) \\ &= 1001. \end{aligned}$$

Since $1001 \geq 10^3$, it follows that

$$(1.001)^n \geq 10^3$$

whenever $n \geq 10^6$.

4. Prove by induction that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution. Let $p(n)$ be the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The induction basis $p(1)$ is

$$1 = \frac{1(1+1)}{2},$$

which is true. Let $n \in \mathbb{N}$ and suppose that $p(n)$ holds. We must show that $p(n+1)$ also holds:

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Hence $p(n+1)$ is true whenever $p(n)$ is true. By induction, $p(n)$ holds for every $n \geq 1$.

5. Prove by induction that

$$\sum_{k=1}^n 4k = 2n(n+1)$$

for every $n \geq 1$.

Solution. Let $p(n)$ be the predicate

$$\sum_{k=1}^n 4k = 2n(n+1).$$

Then $p(1)$ is the statement

$$\sum_{k=1}^1 4k = 2 \cdot 1 \cdot (1+1),$$

which is true, since both sides equal 4. Now let $n \in \mathbb{N}$ with $n \geq 1$ and suppose that $p(n)$ holds. Then

$$\begin{aligned} \sum_{k=1}^{n+1} 4k &= \sum_{k=1}^n 4k + 4(n+1) \\ &= 2n(n+1) + 4(n+1) \\ &= 2(n^2 + 3n + 2) \\ &= 2(n+1)(n+2) \\ &= 2(n+1)((n+1)+1). \end{aligned}$$

It follows that $p(n) \implies p(n+1)$. By mathematical induction, $p(n)$ holds for every $n \geq 1$.

6. Prove by induction that every natural number is even or odd.

Solution. **Basis.** 0 is even since $0 = 2 \cdot 0$. Hence “0 is even or odd” is true.

Inductive step. Let $n \geq 0$ be a natural number and assume n is even or n is odd. If n is even then $n = 2k$ for some natural number k , so $n+1 = 2k+1$ is odd. If n is odd then $n = 2k'+1$ for some natural number k' , so $n+1 = 2(k'+1)$ is even. Thus $n+1$ is even or odd. By the principle of mathematical induction the statement holds for every natural number.