

UNIVERSITY OF PUERTO RICO - RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

ANALYTIC GEOMETRY – PRACTICE QUESTIONS

Instructions: *Show all work to obtain credit.*

Last names: _____ First Name: _____

Student # : _____ Professor: Philip Pennance Section: _____

1. Halle una ecuación para la reflexión de la parábola $y = (x + 1)^2$ en la recta $x = \pi$.

Solution.

The reflection of a point (x, y) in the vertical line $x = \pi$ is $(2\pi - x, y)$. Substituting $x \mapsto 2\pi - x$ into the given equation yields

$$y = (2\pi - x + 1)^2 = (x - 2\pi - 1)^2.$$

2. Halle el punto $(x, y) \in \mathbb{R}^2$ tal que $(x + y, x - y) = (14, 26)$.

Solution:

$$x + y = 14$$

$$x - y = 26$$

Adding and subtracting yields:

$$x = 20$$

$$y = -6$$

3. Halle una ecuación para la recta que pasa por el punto $(\pi, 3)$ si:

(a) es perpendicular a la recta definida por la ecuación $5x - 3 = 3y$.

(b) es paralela a la recta definida por la ecuación $2x - 3 = y$.

Solution.

The line $5x - 3 = 3y$ is equivalent to $y = \frac{5}{3}x - 1$ and has slope $\frac{5}{3}$. A perpendicular line therefore has slope $-\frac{3}{5}$. The line through $(\pi, 3)$ with this slope is

$$y - 3 = -\frac{3}{5}(x - \pi).$$

The line $2x - 3 = y$ has slope 2. The parallel line through $(\pi, 3)$ is

$$y - 3 = 2(x - \pi).$$

4. Halle el punto $(x, y) \in \mathbb{R}^2$ tal que $(x + y, x - y) = (1, -2)$.

Solution:

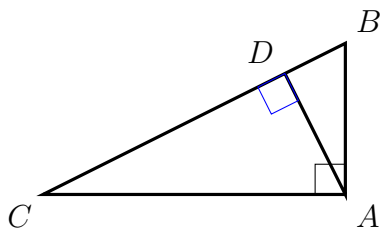
$$\begin{aligned}x + y &= 1 \\x - y &= -2\end{aligned}$$

Adding and subtracting yields:

$$\begin{aligned}x &= -\frac{1}{2} \\y &= \frac{3}{2}\end{aligned}$$

5. Suppose that $A = (0, 0)$, $B = (0, 1)$ and $C = (-2, 0)$. If D is the foot of the perpendicular from A to the side BC , find the coordinates of D .

Solution.



The line ℓ incident to C and B has slope $m = \frac{1}{2}$ and equation $y = \frac{1}{2}x + 1$. Let $D = (a, b)$. Since $D \in \ell$,

$$b = \frac{1}{2}a + 1. \quad (1)$$

The line ℓ' incident to A and D is perpendicular to ℓ . Hence the slope m' of ℓ' satisfies the perpendicularity condition $mm' = -1$. Hence $m' = -2$. Computing m' directly from the slope formula gives $m' = \frac{b}{a}$. Since the slope of a line is constant,

$$\frac{b}{a} = -2. \quad (2)$$

Solving equations (1) and (2) yields $a = -\frac{2}{5}$ and $b = \frac{4}{5}$. Therefore $D = (-\frac{2}{5}, \frac{4}{5})$.

6. Los puntos $A = (5, -2)$, $B = (-3, 4)$, $C = (7, 6)$ son vértices de un triángulo.

(a) Halle la medida de sus lados.

(b) Halle los puntos medios de sus lados. **Solution.**

$$\begin{aligned}AB &= \sqrt{(5 - (-3))^2 + (-2 - 4)^2} = \sqrt{64 + 36} = 10, \\BC &= \sqrt{(-3 - 7)^2 + (4 - 6)^2} = \sqrt{100 + 4} = \sqrt{104} = 2\sqrt{26}, \\AC &= \sqrt{(5 - 7)^2 + (-2 - 6)^2} = \sqrt{4 + 64} = \sqrt{68} = 2\sqrt{17}.\end{aligned}$$

The midpoints are

$$M_{AB} = \left(\frac{5 + (-3)}{2}, \frac{-2 + 4}{2} \right) = (1, 1),$$

$$M_{BC} = \left(\frac{-3 + 7}{2}, \frac{4 + 6}{2} \right) = (2, 5),$$

$$M_{AC} = \left(\frac{5 + 7}{2}, \frac{-2 + 6}{2} \right) = (6, 2).$$

7. Determine si los puntos $(-1, 2)$, $(3, -5)$, $(-5, 8)$ son colineales. **Solution.**

The slope from $(-1, 2)$ to $(3, -5)$ is

$$m_1 = \frac{-5 - 2}{3 - (-1)} = -\frac{7}{4}.$$

The slope from $(-1, 2)$ to $(-5, 8)$ is

$$m_2 = \frac{8 - 2}{-5 - (-1)} = -\frac{3}{2}.$$

Since $m_1 \neq m_2$, the three points are not collinear.

8. En cada ejercicio encuentra una función que cumple la condición dada.

(a) Su gráfica es simétrica con respecto a la línea $x = 5$.

(b) Su gráfica es simétrica con respecto a la línea $y = x$.

(c) Su gráfica pasa por el punto $(5, -7)$.

Solution.

(a) $y = (x - 5)^2$ (or $y = |x - 5|$).

(b) $y = x$ (or $y = \frac{1}{x}$).

(c) $y = x - 12$ (or $y = -7$).

9. Halle una ecuación para la recta que pasa por el punto $(-2, 3)$ si:

(a) es perpendicular a la recta definida por la ecuación $5x - 3 = 3y$.

Solution:

$$y - 3 = -\frac{3}{5}(x + 2)$$

(b) es paralela a la recta definida por la ecuación $5x - 3 = 3y$.

Solution:

$$y - 3 = \frac{5}{3}(x + 2)$$

10. Una ecuación de la recta incidente al punto $(1, 2)$ y que es paralela a la recta con ecuación $2y + 4x + 6 = 0$ es:

(a) $y = -2x + 4$	(c) $y = -2x + 5$	(e) Ninguna de las anteriores.
(b) $y = 4x - 2$	(d) $y = 4x - 7$	

Solution.

$2y + 4x + 6 = 0$ is equivalent to $y = -2x - 3$, which has slope -2 . The parallel line through $(1, 2)$ is $y - 2 = -2(x - 1)$, or $y = -2x + 4$. The correct choice is (a).

11. Halle una ecuación para la recta L si:

- (a) pasa por el punto $(-2, 3)$ y es perpendicular a la recta definida por la ecuación $5x + \pi = 3y$;
- (b) pasa por el punto $(-2, 3)$ y es paralela a la recta definida por la ecuación $5x + \pi = 3y$.

Solution.

$5x + \pi = 3y$ is equivalent to $y = \frac{5}{3}x + \frac{\pi}{3}$ and has slope $\frac{5}{3}$.

- (a) The perpendicular line through $(-2, 3)$ is

$$y - 3 = -\frac{3}{5}(x + 2).$$

- (b) The parallel line through $(-2, 3)$ is

$$y - 3 = \frac{5}{3}(x + 2).$$

12. Write an equation for the perpendicular bisector of the segment from $(-1, \pi)$ to $(\pi, -1)$.

Solution.

Let $P = (-1, \pi)$ and $Q = (\pi, -1)$. The slope of PQ is

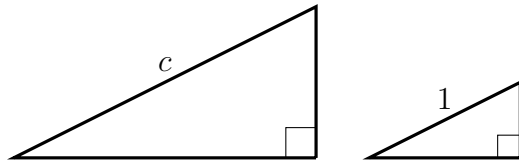
$$m = \frac{(-1) - \pi}{\pi - (-1)} = -1,$$

and hence the perpendicular slope is $m^\perp = 1$. The midpoint M of the segment PQ is the point $M = \left(\frac{-1 + \pi}{2}, \frac{-1 + \pi}{2}\right)$. By definition, the perpendicular bisector is the line incident to M with slope $m^\perp$. An equation for this line is:

$$y - \left(\frac{-1 + \pi}{2}\right) = 1 \cdot \left(x - \frac{-1 + \pi}{2}\right)$$

$$\iff y = x.$$

13. The small triangle has hypotenuse 1 cm and area α cm². The large triangle is similar to the small triangle and has hypotenuse c cm. Find the area of the large triangle in terms of c and α .



Solution.

Let h , b , respectively, be the height and base of the small triangle. Let h' , b' be the height and base of the large triangle. Since the triangles are similar,

$$\frac{b'}{b} = \frac{h'}{h} = \frac{c}{1}.$$

Hence the area of the large triangle is

$$\begin{aligned} \frac{1}{2}b'h' &= \frac{1}{2}(cb)(ch) \\ &= c^2 \frac{1}{2}bh \\ &= c^2 \alpha. \end{aligned}$$

Generally, since area has dimension L^2 , similarity (dilation) by a factor c will increase area by a factor c^2 . It follows that the area of the large triangle is αc^2 .

14. A swimming pool has three water pumps. Working alone, the pumps take 1 hr, 2 hrs, 4 hrs respectively to fill the pool. Find the time required to fill the pool with all 3 pumps working simultaneously.

Solution.

Assume that the volume V of water emitted by a pump is directly proportional to time t . This means that there exists a constitutive constant r such that

$$V = rt.$$

In the (V, t) plane, this is the equation of a straight line through the origin with slope r . The constant r is typically called the *rate constant* of the pump. Let r_1 , r_2 , and r_3 be the rate constants of the three pumps and let V_p be the volume of the pool. From

the information given

$$V_p = r_1 \cdot 1$$

$$V_p = r_2 \cdot 2$$

$$V_p = r_3 \cdot 4$$

Hence

$$r_1 = \frac{V_p}{1}, \quad r_2 = \frac{V_p}{2}, \quad r_3 = \frac{V_p}{4}.$$

With all three pumps functioning, the rate constant is $r = r_1 + r_2 + r_3$ and so

$$\begin{aligned} V &= (r_1 + r_2 + r_3)t \\ &= \left(\frac{V_p}{1} + \frac{V_p}{2} + \frac{V_p}{4} \right) t. \end{aligned}$$

Hence, if τ is the time to fill the pool,

$$V_p = \left(\frac{V_p}{1} + \frac{V_p}{2} + \frac{V_p}{4} \right) \tau.$$

From which it follows that

$$\tau = \frac{1}{\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} \right)} = \frac{4}{7} \text{ hours.}$$

15. Dibuje una gráfica de los conjuntos siguientes.

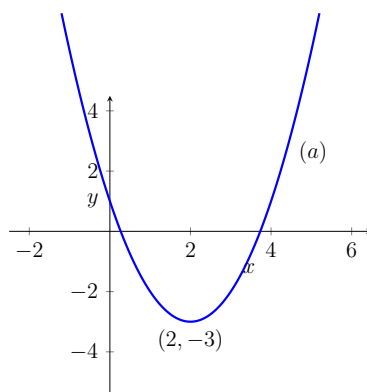
(a) $\{(x, y) \in \mathbb{R}^2 : y = (x - 2)^2 - 3\}$

(b) $\{(x, y) \in \mathbb{R}^2 : y + 2 = -\sqrt{1 - x}\}$

(c) $\{(x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} = 1\}$

Solution.

(a) The set is the graph of $y = (x - 2)^2 - 3$, a parabola opening upward with vertex $(2, -3)$ and axis of symmetry $x = 2$. The y -intercept is $(0, 1)$. The x -intercepts satisfy $(x - 2)^2 = 3$, so $x = 2 \pm \sqrt{3}$.

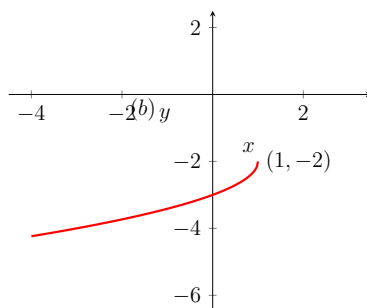


Solution.

(b) The relation $y + 2 = -\sqrt{1 - x}$ requires $1 - x \geq 0$, hence $x \leq 1$, and also $y + 2 \leq 0$, hence $y \leq -2$. Squaring gives

$$(y + 2)^2 = 1 - x \iff x = 1 - (y + 2)^2.$$

Thus the set is the lower half (including the vertex) of the leftward parabola with vertex $(1, -2)$. In particular it contains $(1, -2)$ and $(0, -3)$.



Solution.

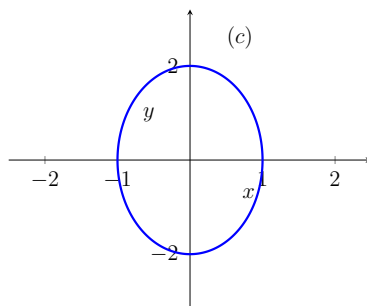
(c) The equation

$$x^2 + \frac{y^2}{4} = 1$$

is the ellipse

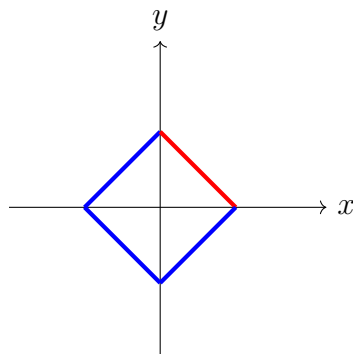
$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

centered at the origin, with vertices $(\pm 1, 0)$ and $(0, \pm 2)$.



16. Sketch the set of points (x, y) in the plane which satisfy $|x| + |y| = 1$. Justify your answer.

Solution.



In the first quadrant both x and y are positive so that $|x| = x$ and $|y| = y$. Hence the intersection of the required set with the first quadrant is the line segment with equation $x + y = 1$ (drawn in red). The other quadrants are handled in a similar manner.

17. The laws of reflection state that an object and its reflection are equidistant from the mirror and that the line connecting an object and its mirror image is perpendicular to the mirror.

(a) Find (in terms of α and β) the reflection of the point (α, β) in the line $x = \pi$.

Solution.

The line $x = \pi$ is vertical and so the line connecting the point $P = (\alpha, \beta)$ with its reflection $P' = (\alpha', \beta')$ must be horizon-

tal. This implies $\beta' = \beta$. To find α' notice that the midpoint of the segment PP' must lie on the line $x = \pi$. Hence, by the midpoint formula $\frac{\alpha + \alpha'}{2} = \pi$.

It follows that $\alpha' = 2\pi - \alpha$ and so the reflection P' of (α, β) is the point $(2\pi - \alpha, \beta)$.

- (b) Find (in terms of a and b) the reflection of the point $P = (a, b)$ in the line $y = x$.

Solution.

Let $P' = (a', b')$ be the reflection of P . The midpoint $(\frac{a'+a}{2}, \frac{b'+b}{2})$ of PP' must lie on the line $y = x$. Hence,

$$\frac{a' + a}{2} = \frac{b' + b}{2}$$

or, equivalently

$$b' - a' = a - b. \quad (3)$$

By the laws of reflection, the segment PP' must be perpendicular to the line

$y = x$. Hence, using the theorem on perpendicular slopes, the slope of PP' must be -1 . Hence $\frac{b'-b}{a'-a} = -1$ and so

$$b' + a' = a + b \quad (4)$$

Solving (3) and (4) yields $a' = b$ and $b' = a$. Hence, the reflection of the point $P = (a, b)$ in the line $y = x$ is the point $P' = (b, a)$.

18. Halle la reflexión de $(\pi^2, 4)$ en la recta $y = -5$.

Solution.

The line $y = -5$ is horizontal, so the segment joining a point to its reflection is vertical. Let $P = (\pi^2, 4)$ and let $P' = (\pi^2, y')$ be the reflection. The midpoint of PP' lies on $y = -5$:

$$\frac{4 + y'}{2} = -5 \iff y' = -14.$$

Hence the reflection is $(\pi^2, -14)$.

19. Halle la reflexión de $(\pi^2, \sqrt{17})$ en la recta $x = 5$.

Solution.

The line $x = 5$ is vertical, so the segment joining a point to its reflection is horizontal. Let $P = (\pi^2, \sqrt{17})$ and let $P' = (x', \sqrt{17})$ be the reflection. The midpoint of PP' lies on $x = 5$:

$$\frac{\pi^2 + x'}{2} = 5 \iff x' = 10 - \pi^2.$$

Hence the reflection is $(10 - \pi^2, \sqrt{17})$.

20. Una ecuación para la reflexión de la semi-parábola $x = \sqrt{y}$ en la recta $y = x$.

Solution.

Reflection in the line $y = x$ interchanges the coordinates x and y . The graph of $x = \sqrt{y}$ is therefore sent to the graph of $y = \sqrt{x}$. An equation for the reflected curve is

$$y = \sqrt{x}.$$

Equivalently, $x = y^2$ with $y \geq 0$.

21. Let $P = (x, y)$ be a point in the plane. If the distance between P and $(0, \frac{1}{4})$ is the same as the perpendicular distance between P and the line $y = -\frac{1}{4}$, find and simplify the relation between x and y .

Solution.

The distance d between P and $(0, \frac{1}{4})$ is

$$d = \sqrt{x^2 + (y - \frac{1}{4})^2}.$$

The perpendicular distance $d^\perp$ between P and the line $y = -\frac{1}{4}$ is $|y + \frac{1}{4}|$.

$$\begin{aligned} d^\perp = d &\iff \sqrt{(y + \frac{1}{4})^2} = \sqrt{x^2 + (y - \frac{1}{4})^2} \\ &\iff (y + \frac{1}{4})^2 = x^2 + (y - \frac{1}{4})^2 \\ &\iff y^2 + \frac{1}{2}y + \frac{1}{16} = x^2 + y^2 - \frac{1}{2}y + \frac{1}{16} \\ &\iff y = x^2. \end{aligned}$$

22. Let ℓ be a line with $(0, 0) \notin \ell$. Which of the following alternatives are sufficient to guarantee that the slope of ℓ be negative? Justify your answers.

(a) The vertical intercept is twice the horizontal.

(b) The product of both intercepts is positive.

(c) $(a, b) \in \ell$ and $(r, s) \in \ell$ and $(a - r)(b - s) < 0$

Solution.

Let a, b be the horizontal and vertical intercepts respectively. Then $(a, 0) \in \ell$ and $(0, b) \in \ell$ and so ℓ has slope $m = -\frac{b}{a}$. If the vertical intercept is twice the horizontal, then $b = 2a$ and it follows that $m = -2 < 0$. Hence condition (a) is a sufficient condition that ℓ has a negative slope.

If $ab > 0$ then division by a^2 yields $\frac{b}{a} > 0$ and it follows that $m = -\frac{b}{a} < 0$. Hence condition (b) is sufficient for a negative slope.

Suppose that $(a, b) \in \ell$ and $(r, s) \in \ell$. Then $m = \frac{b-s}{a-r}$. If $(a - r)(b - s) < 0$, it follows that $a - r$ and $b - s$ have opposite signs and so $m < 0$. Hence condition (c) is also sufficient for a negative slope.

23. Find the perpendicular distance of the point $(2, -1)$ from the line $3x - 4y = 3$. (Hint: non-vertical lines are perpendicular if and only if the product of the slopes is -1 . No credit will be given for merely substituting in a formula.)

Solution.

The line $\ell : 3x - 4y = 3$ has slope $m = \frac{3}{4}$. The slope of any line perpendicular to ℓ must be $-\frac{4}{3}$. It follows that the perpendicular line $\ell^\perp$ incident to $(2, -1)$ has equation $y + 1 = -\frac{4}{3}(x - 2)$. Solving the two equations simultaneously shows that ℓ and $\ell^\perp$ intersect at the point $(\frac{29}{25}, \frac{3}{25})$. By the distance formula, the distance d between $(2, -1)$ and $(\frac{29}{25}, \frac{3}{25})$ is

$$d = \sqrt{\left(2 - \frac{29}{25}\right)^2 + \left(-1 - \frac{3}{25}\right)^2} = \frac{7}{5}.$$

24. Use the concepts of translation, reflection and dilatation (zoom) to sketch the graphs of

(a) $y = |x - 2| - 2$

(b) $y = -|x - 2| - 2$

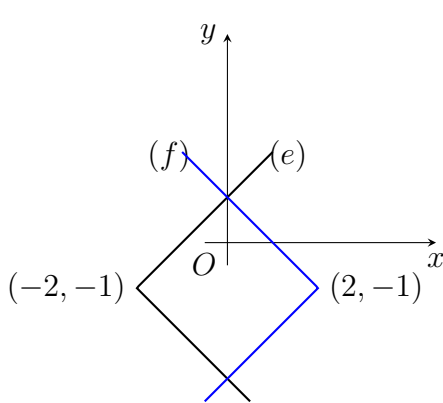
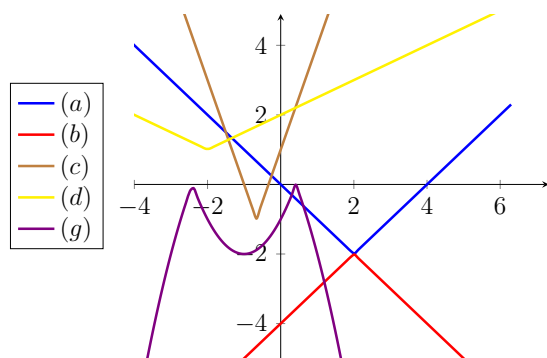
(c) $y = |3x + 2| - 1$

(d) $2y = |x + 2| + 2$

(e) $x = |y + 1| - 2$

(f) $-x = |y + 1| - 2$

(g) $y = -|-x^2 - 2x + 1|$



25. Let a be a real number and define a^+ by:

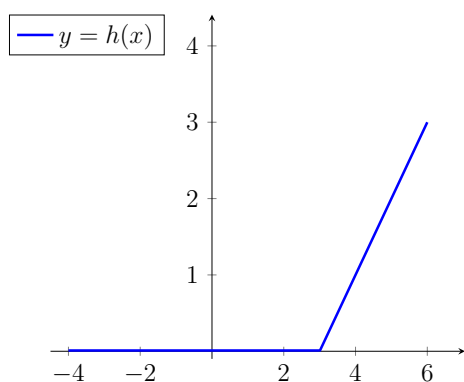
$$a^+ = \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

Draw the graphs of the following functions:

(a) $h(x) = (x - 3)^+$.

Solution.

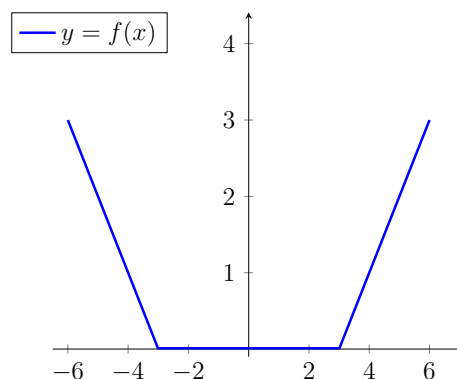
$$h(x) = \begin{cases} x - 3 & \text{if } x \geq 3, \\ 0 & \text{if } x < 3. \end{cases}$$



(b) $f(x) = (|x| - 3)^+$.

Solution.

$$f(x) = \begin{cases} -x - 3 & \text{if } x \leq -3, \\ 0 & \text{if } -3 < x < 3, \\ x - 3 & \text{if } x \geq 3. \end{cases}$$



26. Halle la distancia d entre el punto (a, b) y $(2, -3)$ si $a^2 + b^2 - 4a + 6b - 1 = 0$.

Solution.

$$\begin{aligned} a^2 + b^2 - 4a + 6b - 1 = 0 &\iff (a^2 - 4a) + (b^2 + 6b) = 1 \\ &\iff (a - 2)^2 - 4 + (b + 3)^2 - 9 = 1 \\ &\iff (a - 2)^2 + (b + 3)^2 = 14 \\ &\iff d[(a, b), (2, -3)] = \sqrt{14}. \end{aligned}$$

27. Un círculo tiene la ecuación $x^2 + y^2 + 4x + 2y = 21$. Halle el centro y el radio.

Solution.

$$\begin{aligned} x^2 + y^2 + 4x + 2y = 21 &\iff (x + 2)^2 - 4 + (y + 1)^2 - 1 = 21 \\ &\iff (x + 2)^2 + (y + 1)^2 = 26. \end{aligned}$$

By the distance formula, the center is $(-2, -1)$ and the radius is $\sqrt{26}$.

28. Halle una ecuación para el círculo con centro $(4, 6)$ si pasa por el punto medio de $(4, 6)$ y $(8, -6)$.

Solution.

Sea $P = (4, 6)$ y $Q = (8, -6)$. Sea M el punto medio de P y Q :

$$M = \left(\frac{4+8}{2}, \frac{6+(-6)}{2} \right) = (6, 0).$$

The circle has radius

$$r = d(P, M) = \sqrt{(4-6)^2 + (6-0)^2} = \sqrt{40}.$$

An equation for the circle of radius $\sqrt{40}$ and center $(4, 6)$ is

$$(x-4)^2 + (y-6)^2 = 40.$$

29. Halle una ecuación para la traslación 2 unidades hacia la derecha de la gráfica de $x^2 + y^2 = 1$.

Solution.

A translation of 2 units to the right replaces x by $x-2$. An equation for the translated graph is

$$(x-2)^2 + y^2 = 1.$$

30. Halle todo punto (x, y) en el plano que cumple $x^2 + (y-1)^2 = 0$.

Solution.

A sum of squares is zero if and only if each square is zero:

$$x^2 = 0 \quad \text{and} \quad (y-1)^2 = 0 \iff x = 0 \quad \text{and} \quad y = 1.$$

The only point is $(0, 1)$.

31. Un círculo tiene la ecuación $x^2 + y^2 - 2y = 15$. Halle el centro y el radio.

Solution:

$$\begin{aligned} x^2 + y^2 - 2y &= 15 \\ \iff x^2 + (y-1)^2 &= 16 \\ \iff \sqrt{(x-0)^2 + (y-1)^2} &= 4. \end{aligned}$$

Centro: $(0, 1)$ Radio: 4.

32. Sketch the set $\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + (y + 2)^2} = 2\}$

Solution.

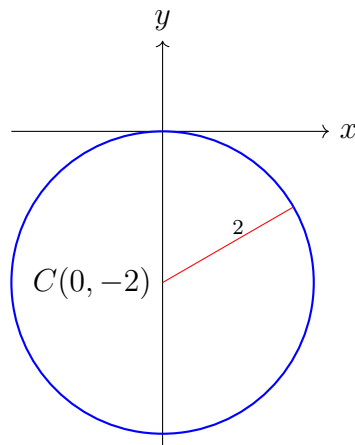
By Pythagoras' theorem, the distance $d(P, C)$ between a point (x, y) and the point $C = (0, -2)$ is

$$d(P, C) = \sqrt{[x - 0]^2 + [y - (-2)]^2}.$$

Thus the given set can be written

$$\{P \in \mathbb{R}^2 : d(P, C) = 2\}.$$

It consists of all points in the plane whose distance from the fixed point C is 2. By definition this is the circle with center C and radius 2.



33. Halle el centro y el radio del círculo $x^2 + y^2 - 2\pi x + 2\sqrt{3}y = 10\sqrt{\pi}$.

Solution.

$$\begin{aligned} x^2 + y^2 - 2\pi x + 2\sqrt{3}y = 10\sqrt{\pi} &\iff (x - \pi)^2 + (y + \sqrt{3})^2 - \pi^2 - 3 = 10\sqrt{\pi} \\ &\iff (x - \pi)^2 + (y + \sqrt{3})^2 = \pi^2 + 3 + 10\sqrt{\pi}. \end{aligned}$$

By the distance formula, the center is $(\pi, -\sqrt{3})$ and the radius is $\sqrt{\pi^2 + 3 + 10\sqrt{\pi}}$.

34. El centro C y el radio r del círculo con ecuación $x^2 + y^2 - 4x + 6y - 1 = 0$ son:

(a) $C = (-2, 3), r = 14$

(b) $C = (-2, 3), r = \sqrt{14}$

(c) $C = (2, -3), r = 14$

(d) $C = (2, -3), r = \sqrt{14}$

(e) $C = (-4, 6), r = 0$

Solution.

Complete the square:

$$\begin{aligned} x^2 + y^2 - 4x + 6y - 1 = 0 &\iff (x^2 - 4x) + (y^2 + 6y) = 1 \\ &\iff (x - 2)^2 - 4 + (y + 3)^2 - 9 = 1 \\ &\iff (x - 2)^2 + (y + 3)^2 = 14. \end{aligned}$$

The center is $C = (2, -3)$ and the radius is $r = \sqrt{14}$. The correct choice is (d).

35. (a) Demuestre que el punto $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ se encuentra en el círculo

$$(x+1)^2 + y^2 = 1.$$

- (b) Halle una ecuación para la recta tangente al círculo $(x+1)^2 + y^2 = 1$ en el punto $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Solution.

(a) Substitute the given point:

$$\left(-\frac{1}{2} + 1\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{1}{4} + \frac{3}{4} = 1.$$

The point lies on the circle.

(b) The circle has center $(-1, 0)$ and radius 1. The radius from $(-1, 0)$ to $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ has slope

$$m = \frac{\frac{\sqrt{3}}{2} - 0}{-\frac{1}{2} - (-1)} = \sqrt{3}.$$

The tangent is perpendicular to this radius, so its slope is $-\frac{1}{\sqrt{3}}$. An equation of the tangent line is

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}} \left(x + \frac{1}{2}\right),$$

which simplifies to

$$x + \sqrt{3}y = 1.$$

36. Halle el foco y la directriz de la parábola definida por la ecuación $4y = (x-1)^2 + 3$.

Solution:

$$4y = (x-1)^2 + 3 \iff y = \frac{1}{4}(x-1)^2 + \frac{3}{4}.$$

Here $1/(4p) = 1/4$, so $p = 1$. The vertex is $(1, 3/4)$. Focus: $(1, 3/4 + 1) = (1, 7/4)$.

Directrix: $y = 3/4 - 1 = -1/4$.

37. Halle el vértice, el foco, el eje de simetría, la directriz y los interceptos de la parábola $y = x^2 - 6x - 13$.

Solution.

$$\begin{aligned} y = x^2 - 6x - 13 &\iff y = (x-3)^2 - 9 - 13 \\ &\iff y = (x-3)^2 - 22. \end{aligned}$$

Vertex: $(3, -22)$; axis of symmetry: $x = 3$. Since $1/(4p) = 1$, one has $p = 1/4$. Focus: $(3, -22 + \frac{1}{4}) = (3, -\frac{87}{4})$. Directrix: $y = -22 - \frac{1}{4} = -\frac{89}{4}$. Horizontal intercepts: $x = 3 \pm \sqrt{22}$. Vertical intercept: $(0, -13)$.

38. Defina *parábola* con foco F y directriz D .

Solution.

Let D be a fixed line in the plane and let F be a point in the plane with $F \notin D$. The *parabola with focus F and directrix D* is the set of all points P in the plane such that the distances $d(P, F)$ and $d(P, D)$ are equal.

Remark. By $d(P, D)$ is meant the shortest distance from P to the line D .

39. Use la definición para hallar una ecuación para la parábola con foco $(3, 2)$ y directriz $y = -2$. **Solution.** Let $P = (x, y)$ be any point on the parabola. Let F be the point $(3, 2)$ and L the line $y = -2$.

$$\begin{aligned} d(P, F) = d(P, L) &\iff \sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(y+2)^2} \\ &\iff (x-3)^2 + y^2 - 4y + 4 = y^2 + 4y + 4 \\ &\iff 8y = (x-3)^2 \\ &\iff y = \frac{1}{8}(x-3)^2. \end{aligned}$$

40. Escriba una ecuación para la línea que pasa a través del punto (α, β) y es perpendicular a la línea $y = x$. Muestre que la distancia de (α, β) a $y = x$ es $\frac{|\alpha - \beta|}{\sqrt{2}}$. Use esto para escribir una ecuación para la parábola con foco $(1, 0)$ y directriz $y = x$.

Solution.

The line $y = x$ has slope 1. A perpendicular line must have slope -1 . Hence the perpendicular line incident to (α, β) has equation $y - \beta = -(x - \alpha)$ or equivalently $y = -x + (\beta + \alpha)$. This line intersects $y = x$ at the point $(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2})$. The distance of (α, β) from the point of intersection is

$$\sqrt{\left(\frac{\alpha - \beta}{2}\right)^2 + \left(\frac{\alpha - \beta}{2}\right)^2} = \frac{|\alpha - \beta|}{\sqrt{2}}.$$

Finally, if $P = (x, y)$ is a point of the parabola with focus $F = (1, 0)$ and directrix the line $L: y = x$, then $d(P, F) = d(P, L)$. In coordinates this is equivalent to

$$\sqrt{(x-1)^2 + y^2} = \frac{|x-y|}{\sqrt{2}},$$

or equivalently

$$2((x-1)^2 + y^2) = (x-y)^2.$$

41. Halle una ecuación de la línea recta paralela a la recta $-2x + 3y = 4$ si pasa por el foco de la parábola $y = x^2$.

Solution.

The parabola $y = x^2$ has focus $F = (0, \frac{1}{4})$. The given line has slope $\frac{2}{3}$. Hence the parallel line incident to F has equation

$$y - \frac{1}{4} = \frac{2}{3}x.$$

42. Write an expression for the perpendicular distance from a point $P = (x, y)$ to the horizontal line $y = -\frac{1}{4}$.

Solution.

The vertical distance $d^\perp$ from P to the line $y = -\frac{1}{4}$ is the same as the distance between (x, y) and $(x, -\frac{1}{4})$. By Pythagoras' theorem

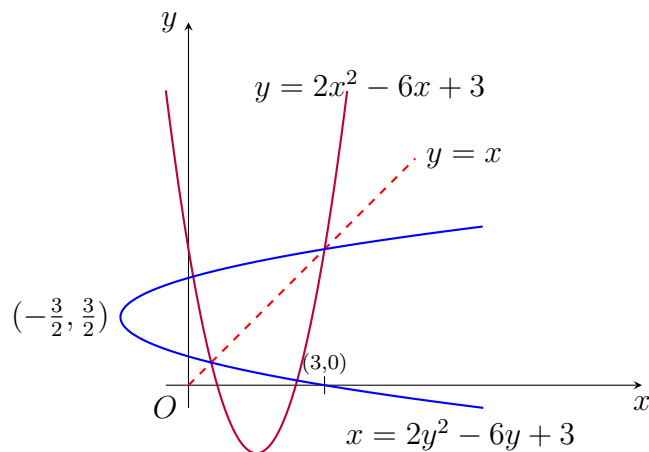
$$\begin{aligned} d^\perp &= \sqrt{[x - x]^2 + [y - (-\frac{1}{4})]^2} \\ &= \sqrt{[y + \frac{1}{4}]^2} \\ &= |y + \frac{1}{4}|. \end{aligned}$$

43. Halle el vértice, el eje de simetría, foco, directriz, los cortes en los ejes y trace la gráfica de la parábola definida por $x = 2y^2 - 6y + 3$

Solution.

$$\begin{aligned} x = 2y^2 - 6y + 3 &\iff x = 2(y^2 - 3y) + 3 \\ &\iff x = 2\left[(y - \frac{3}{2})^2 - \frac{9}{4}\right] + 3 \\ &\iff x = 2(y - \frac{3}{2})^2 - \frac{3}{2}. \end{aligned}$$

The graph is obtained by reflecting $y = 2x^2$ in the line $y = x$ and then translating by the vector $(-\frac{3}{2}, \frac{3}{2})$.



The vertex is $(-\frac{3}{2}, \frac{3}{2})$. The axis of symmetry is the line $y = \frac{3}{2}$. The x -intercept, obtained by putting $y = 0$, is $x = 3$. The y -intercepts, obtained by putting $x = 0$, are $y = \frac{3}{2} \pm \frac{\sqrt{3}}{2}$.

The focus of $y = 2x^2$ is $(0, \frac{1}{8})$. Reflecting this point in the line $y = x$ gives $(\frac{1}{8}, 0)$. Translating yields the focus

$$F = (\frac{1}{8}, 0) + (-\frac{3}{2}, \frac{3}{2}) = (-\frac{11}{8}, \frac{3}{2}).$$

The directrix of $y = 2x^2$ is $y = -\frac{1}{8}$. Reflecting in $y = x$ and translating yields

$$D = \{(x, y) : x = -\frac{13}{8}\}.$$

44. Halle el vértice, el eje de simetría, foco, directriz, los cortes en los ejes y trace la gráfica de la semi-parábola

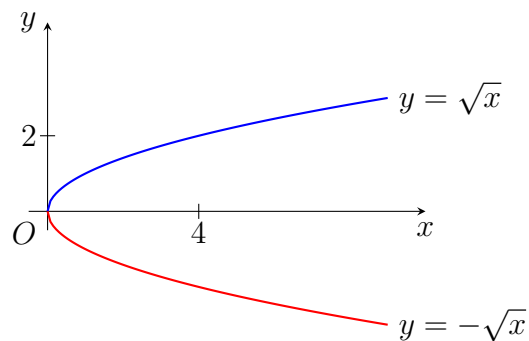
$$\{(x, y) \in \mathbb{R}^2 : y = \sqrt{x}\}$$

Solution.

Let P be the parabola $P = \{(x, y) : x = y^2\}$. Then P is a parabola with focus $(\frac{1}{4}, 0)$ and directrix the vertical line $x = -\frac{1}{4}$. If $P^+ = \{(x, y) \in \mathbb{R}^2 : y = \sqrt{x}\}$, then

$$\begin{aligned} (x, y) \in P^+ &\implies y = \sqrt{x} \\ &\implies x = y^2 \\ &\implies (x, y) \in P. \end{aligned}$$

Hence $P^+ \subseteq P$. Since $\sqrt{x}$ is nonnegative, P^+ is precisely the upper half of the parabola P , including the origin.



Similarly, if $P^- = \{(x, y) \in \mathbb{R}^2 : y = -\sqrt{x}\}$, then P^- is precisely the lower half of the parabola P , including the origin.

45. The laws of reflection state that an object and its reflection are equidistant from the mirror and that the line connecting an object and its mirror image is perpendicular to the mirror. Find (in terms of α and β) the reflection of the point (α, β) in the line $y = -x$.

Solution.

Let $P = (\alpha, \beta)$ and let $P' = (\alpha', \beta')$ be the reflection of P in the line $y = -x$. The line $y = -x$ has slope -1 , so the segment PP' is perpendicular to the mirror if and only if its slope is 1:

$$\frac{\beta' - \beta}{\alpha' - \alpha} = 1 \iff \beta' - \alpha' = \beta - \alpha. \quad (5)$$

The midpoint of PP' must lie on $y = -x$, so

$$\begin{aligned} \frac{\beta + \beta'}{2} &= -\frac{\alpha + \alpha'}{2} \iff \alpha' + \beta' = -(\alpha + \beta). \\ \alpha' + \beta' &= -\alpha - \beta. \end{aligned} \quad (6)$$

From (5), $\beta' = \alpha' + \beta - \alpha$. Substitute into (6):

$$\begin{aligned} \alpha' + (\alpha' + \beta - \alpha) &= -\alpha - \beta \\ 2\alpha' + \beta - \alpha &= -\alpha - \beta \\ 2\alpha' &= -2\beta \\ \alpha' &= -\beta. \end{aligned}$$

Then $\beta' = -\beta + \beta - \alpha = -\alpha$. Therefore the reflection of (α, β) in the line $y = -x$ is the point $(-\beta, -\alpha)$.

46. Halle constantes β, γ such that $x^2 + 3x - 8 = (x - \beta)^2 - \gamma^2$.

Solution.

$$\begin{aligned} x^2 + 3x - 8 &= \left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8 \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{41}{4} \\ &= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{41}{4}}\right)^2. \end{aligned}$$

Por lo tanto $\beta = -\frac{3}{2}$ y $\gamma = \sqrt{\frac{41}{4}}$.

47. Halle las dos raíces complejas de $w = 3 + 4i$.

Solution.

If $z = a + ib$ satisfies $z^2 = w$, then

$$\begin{aligned} |z|^2 &= |w| \\ \operatorname{Re} z^2 &= \operatorname{Re} w \\ \operatorname{Im} z^2 &= \operatorname{Im} w. \end{aligned}$$

Writing these equations in terms of coordinates gives

$$a^2 + b^2 = 5 \tag{7}$$

$$a^2 - b^2 = 3 \tag{8}$$

$$2ab = 4. \tag{9}$$

From (7) and (8) it follows that $a \in \{2, -2\}$ and $b \in \{1, -1\}$. From (9) it is deduced that a and b must have the same sign. The only possibilities are $z = 2+i$ and $z = -2-i$. In either case it is easy to check that $z^2 = w$.

48. Find the perpendicular distance of the point $(2, -1)$ from the line $3x - 4y = 3$. (Hint: non vertical lines are perpendicular if and only if the product of the slopes is -1 . No credit will be given for merely substituting in a formula.)

Solution.

The line ℓ with equation $3x - 4y = 3$ has slope $m = \frac{3}{4}$. The slope of any line perpendicular to ℓ must be $-\frac{4}{3}$. It follows that the perpendicular line $\ell^\perp$ incident to $(2, -1)$ has equation $y + 1 = -\frac{4}{3}(x - 2)$. Solving the two equations simultaneously shows that ℓ and $\ell^\perp$ intersect at the point $(\frac{29}{25}, \frac{3}{25})$. By the distance formula, the distance d between

$(2, -1)$ and $(\frac{29}{25}, \frac{3}{25})$ is

$$\begin{aligned} d &= \sqrt{\left(2 - \frac{29}{25}\right)^2 + \left(-1 - \frac{3}{25}\right)^2} \\ &= \sqrt{\left(\frac{21}{25}\right)^2 + \left(\frac{28}{25}\right)^2} \\ &= \sqrt{\left(\frac{7}{25}\right)^2 (3^2 + 4^2)} \\ &= \frac{7}{5} \sqrt{3^2 + 4^2} \\ &= \frac{7}{5}. \end{aligned}$$

49. Let ℓ be a line with $(0, 0) \notin \ell$. Which of the following alternatives are sufficient to guarantee that the slope of ℓ be negative? Justify your answers.
- (a) The vertical intercept is twice the horizontal.
 - (b) The product of both intercepts is positive.
 - (c) $(a, b) \in \ell$ and $(r, s) \in \ell$ and $(a - r)(b - s) < 0$

Solution.

Let a, b be the horizontal and vertical intercepts respectively. Then $(a, 0) \in \ell$ and $(0, b) \in \ell$ and so ℓ has slope $m = -\frac{b}{a}$. If the vertical intercept is twice the horizontal, then $b = 2a$ and it follows that $m = -2 < 0$. Hence condition (a) is a sufficient condition that ℓ has a negative slope.

If $ab > 0$ then division by a^2 yields $\frac{b}{a} > 0$ and it follows that $m = -\frac{b}{a} < 0$. Hence condition (b) is sufficient for a negative slope.

Suppose that $(a, b) \in \ell$ and $(r, s) \in \ell$. Then, $m = \frac{b-s}{a-r}$. If $(a - r)(b - s) < 0$, it follows that $a - r$ and $b - s$ have opposite signs and so $m < 0$. Hence condition (c) is also sufficient for a negative slope.

50. Write an equation for the perpendicular bisector of the segment from $(-1, \pi)$ to $(\pi, -1)$

Solution.

Let $P = (-1, \pi)$ and $Q = (\pi, -1)$. The slope of PQ is

$$m = \frac{(-1) - \pi}{\pi - (-1)} = -1.$$

and, hence, the perpendicular slope is $m^\perp = 1$. The midpoint M of the segment PQ is the point $M = \left(\frac{-1 + \pi}{2}, \frac{-1 + \pi}{2}\right)$. By definition, the perpendicular bisector is the

line incident to M with slope $m^\perp$. An equation for this line is:

$$y - \left(\frac{-1 + \pi}{2} \right) = 1 \cdot \left(x - \frac{-1 + \pi}{2} \right) \\ \Longleftrightarrow y = x.$$

51. Halle una ecuación para la reflexión de la parabola $y = (x + 1)^2$ en la recta $x = \pi$.

Solution.

$$y = (x - 2\pi - 1)^2$$