1. Let \( G \) be an e-labelled graph with edge set \( E(G) \). A graph \( F \) is called an \textit{abstract dual} or \textit{matroid dual} of \( G \) if there exists a bijection \( \varepsilon : E(G) \rightarrow E(F) \) such that:

\[ C \in C(G) \iff \varepsilon(C) \in C^*(F) \]

By edge relabelling, the bijection \( \varepsilon \) can be taken to be the identity so that \( E(F) = E(G) \) and \( C(G) = C^*(F) \).

2. There may exist two abstract duals \( F \) and \( F' \) with the same bijection \( \varepsilon \).

3. Let \( F, G \) be e-labelled graphs with \( E(G) = E(F) \). A bijection \( \varepsilon : E(G) \rightarrow E(F) \) such that:

\[ C \in C(G) \iff \varepsilon(C) \in C(F) \]

is called a \textit{cycle isomorphism}.

4. The following are equivalent:

(a) Cycle isomorphism: \( C(G) = C(F) \)

(b) Cocycle isomorphism \( C^*(G) = C^*(F) \)

(c) Tree isomorphism \( T(G) = T(F) \)

(d) Cotree isomorphism \( T^*(G) = T^*(F) \)

(e) Forest isomorphism \( F(G) = F(F) \)

(f) Co-forest isomorphism \( F^*(G) = F^*(F) \)

5. Since

\[ \operatorname{Min} \perp C(G) = C^*(G) \quad \text{and} \quad \operatorname{Min} \perp C^*(G) = C(G) \]

the following are equivalent:

(a) Abstract duality \( C(G) = C^*(F) \)

(b) Abstract duality* \( C^*(G) = C(F) \)

(c) Tree duality \( T(G) = T^*(F) \)

(d) Co-tree duality \( T^*(G) = T(F) \)

(e) Forest duality \( F(G) = F^*(F) \)

(f) Co-forest duality \( F^*(G) = F(F) \)

6. If \( F \) is the abstract dual of \( G \) then

(a) \( G \) is the abstract dual of \( F \)

(b) \( T \) is a spanning tree of \( G \) if and only if \( E \setminus T \) is a spanning tree of \( F \).

7. If \( G \) is a plane graph, the geometric dual of \( G \) is also an abstract dual.

8. \( K_{33} \) and \( K_5 \) have no abstract dual.

9. The class of graphs with an abstract dual is closed under the operations of subdivision and taking a subgraph.

10. Theorem [H. Whitney]

The following are equivalent:

(a) \( G \) has an abstract dual.

(b) \( G \) is planar.

Proof: If \( G \) is planar, it has a geometric dual and hence an abstract dual. Conversely, if \( G \) is non planar, it contains a subgraph which is subdivision of \( K_{33} \) or \( K_5 \). This is impossible in a graph with an abstract dual.

\[ \text{Source: http://pennance.us} \]