The Limit of a Sequence (Brief Summary) ¹

1. Definition. A real number \( L \) is a limit of a sequence of real numbers if every open interval containing \( L \) contains all but a finite number of terms of the sequence.

2. Claim. A sequence can have at most one limit.

Proof: Suppose \( L \) and \( L' \) be limits of a sequence. Then there exists an open interval \( I \) containing \( L \) and an open interval \( I' \) containing \( L' \) and both of these intervals contain all but a finite number of terms of the sequence. If \( L \neq L' \), \( I \) and \( I' \) can be chosen to be disjoint which (exercise) leads to a contradiction. Thus, it must be that \( L = L' \).

3. Notation. If \( L \) is the limit of a sequence \((a_n)\) we write

\[
\lim_{n \to \infty} a_n = L
\]

and we say that the sequence converges to \( L \) or that the sequence is convergent. A sequence with no limit is called divergent.

4. Remark. The definition of limit gives precise meaning to the rather vague phrase “\( a_n \) approaches \( L \) as \( n \) approaches \( \infty \)”. This statement does not serve as the definition of limit because neither the nature of the “approach” nor the concept of \( \infty \) are explained.

5. Lemma. Let \( I \) is any open interval containing \( L \), then within \( I \) can be found an open interval of the form \( |x - L| < \epsilon \)

6. Theorem. Let \((a_n)\) be a sequence. The following are equivalent,

(a) \[ \lim_{n \to \infty} a_n = L \]

(b) For every \( \epsilon > 0 \), all but a finite number of terms of the sequence are contained in the interval with center \( L \) and radius \( \epsilon \).

(c) \[ \forall \epsilon > 0, \exists N \text{ such that } \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon \]

7. Example: Let \((a_n)\) be the sequence given by

\[ a_n = \frac{n}{n + 1}. \]

Show that \[ \lim_{n \to \infty} a_n = 1. \]

Solution: Let \( \epsilon > 0 \) be arbitrary. We must find a natural number \( N \) such that \( \forall n \in \mathbb{N} \)

\[ n > N \implies |a_n - 1| < \epsilon \]

whenever \( n > N \). Now

\[ d(a_n, 1) = \left| \frac{n}{n + 1} - 1 \right| = \frac{1}{n + 1}. \]

Moreover, since \( \mathbb{N} \) is unbounded above there exists a natural number \( N \) such that \( Ne > 1 \). It follows that:

\[ n > N \implies n > \frac{1}{\epsilon} \]
\[ \implies \epsilon n > 1 \]
\[ \implies \frac{1}{n} < \epsilon \]
\[ \implies \frac{1}{n + 1} < \epsilon \]
\[ \implies d(a_n, 1) < \epsilon \]


² Notice that the value of \( N \) will depend on the value of \( \epsilon \). The smaller the value of \( \epsilon \) the larger \( N \) must be in order that the open interval determined by equation (1) contains all terms after the \( N \)’th. Figures 3 and 4 suggest that if \( \epsilon = 0.2 \) then \( N = 5 \) works and if \( \epsilon = 0.1 \) then \( N = 9 \) suffices.
Since $N$ has the required property, we conclude that
\[
\lim_{n \to \infty} \frac{n}{n + 1} = 1
\]

8. Example. Let $(a_n)$ be the constant sequence given by $a_n = 5, n \in \mathbb{N}$. Show that $\lim_{n \to \infty} a_n = 5$.

Solution. Let $\epsilon > 0$. It is easy to find $N$ such that $|a_n - 5| < \epsilon$ whenever $n > N$. Let $N = 24$ (any other natural number will do). Since $|a_n - 5| = 0$, for all $n \in \mathbb{N}$ it is clearly true that $|a_n - 5| < \epsilon$ whenever $n > 24$. It follows that $\lim_{n \to \infty} a_n = 5$.

9. More generally, if $a_n = c$ for all $n \in \mathbb{N}$ then $\lim_{n \to \infty} a_n = c$.

Figure 1. Initial Terms of the Sequence $a_n = \frac{n}{n + 1}$

Figure 2. Graph of the sequence $a_n = \frac{n}{n + 1}$

Figure 3. $n > 5 \Rightarrow |a_n - 1| < 0.2.$
10. Example: Let \((a_n)\) be the sequence given by \(a_n = \frac{1}{n-1}\), \(n > 1\). Show using the definition of the limit of a sequence that
\[
\lim_{n \to \infty} a_n = 0.
\]
Solution. Let \(\epsilon > 0\) Notice (exercise) that \(\frac{1}{n-1} < \frac{2}{n}\) for all \(n > 2\). Since \(\mathbb{N}\) is unbounded, there exists a natural number \(N > 1\) such that \(N\epsilon > 2\). It follows that:
\[
n > N \Rightarrow n\epsilon > 2 \\
\Rightarrow 2n < \epsilon
\]
Since \(\frac{1}{n-1} < \frac{2}{n}\), it follows that
\[
n > N \Rightarrow \left| \frac{1}{n-1} \right| < \epsilon \\
\Rightarrow \left| \frac{1}{n-1} - 0 \right| < \epsilon \\
\Rightarrow d(a_n, 0) < \epsilon
\]
which proves that
\[
\lim_{n \to \infty} \frac{1}{n-1} = 1
\]

11. The next theorem reduces the calculation of complex limits to simpler ones.

12. Theorem.
Let \((a_n)\) and \((b_n)\) be sequences. If \(\lim_{n \to \infty} a_n = L\) and \(\lim_{n \to \infty} b_n = M\) then:
(a) \(\lim_{n \to \infty} (a_n \pm b_n) = L \pm M\)
(b) \(\lim_{n \to \infty} (a_n \cdot b_n) = L \cdot M\)
(c) If \(M \neq 0\) and \(b_n \neq 0\) for all \(n\) then
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M}.
\]

13. Example. Find the limit of the sequence \((a_n)\) given by
\[a_n = \frac{3}{n}\]
Solution. Notice \(a_n = x_n \cdot y_n\) where \(x_n = 1/n\) and \(y_n = 3\), \(n \in \mathbb{N}\). Now, it is easy to show that \(\lim_{n \to \infty} x_n = 0\) and \(\lim_{n \to \infty} y_n = 3\). Using part (b) of the above theorem yields
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0 \cdot 3 = 0
\]

14. Example. Find the limit of the sequence \((a_n)\) given by
\[a_n = \frac{1}{n^2}\]
Solution. Let \(x_n = 1/n\). Then
\[a_n = x_n \cdot x_n\]
Since \( \lim_{n \to \infty} x_n = 0 \).

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} x_n \cdot \lim_{n \to \infty} x_n = 0 \cdot 0 = 0.
\]

15. Using mathematical induction (exercise) it is now easy to show that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} x_n \cdot \lim_{n \to \infty} x_n = 0 \cdot 0 = 0.
\]

16. Example. Find the limit of the sequence \( x_n = \frac{n^2 + 3n}{3 + n^2} \).

Solution.

\[
x_n = \frac{n^2 + 3n}{3 + n^2} = \frac{n^2 + n^2}{3 + n^2} = 1 + \frac{3}{n^2}.
\]

But

\[
\lim_{n \to \infty} \left( 1 + \frac{3}{n} \right) = \lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{3}{n} = 1 + 0 = 1
\]

and

\[
\lim_{n \to \infty} \left( \frac{3}{n^2} + 1 \right) = \lim_{n \to \infty} \frac{3}{n^2} + \lim_{n \to \infty} 1 = 0 + 1 = 1
\]

Hence

\[
\lim_{n \to \infty} x_n = \frac{\lim_{n \to \infty} \left( 1 + \frac{3}{n} \right)}{\lim_{n \to \infty} \left( \frac{3}{n^2} + 1 \right)} = 1.
\]

17. Definition. A sequence \((a_n)\) is bounded if there exists a number \(M\) such that \(|a_n| \leq M\) for all \(n \in \mathbb{N}\). A sequence which is not bounded is called unbounded.

18. Example. Consider the sequence \(a_n = \frac{n}{n+1}\). Since \(|a_n| \leq 1\) for all \(n \in \mathbb{N}\), this sequence is bounded.

19. Claim. A convergent sequence is bounded.

Proof. Let \((a_n)\) be a convergent sequence with limit \(L\). Then there exists a number \(N\) such the open interval \((L-1, L+1)\) contains all terms \(a_n\) with \(n > N\). It follows that all terms of \(a_n\) are less than or equal in absolute value to the maximum \(M\) of the set

\[
\{|a_1|, |a_2|, \ldots, |a_N|, |L+1|, |L-1|\}
\]

(Exercise: Find a counterexample to show that the converse of this result is false.)

20. Corollary. The sequence \(a_n = n\) has no limit.

Proof. Notice that the sequence \((a_n)\) is unbounded. If it had a limit it would be bounded.

21. Sandwich Theorem

Let \((a_n)\), \((b_n)\) and \((c_n)\) be sequences. If \(a_n \leq b_n \leq c_n\) for all \(n \in \mathbb{N}\) and the sequences \((a_n)\) and \((c_n)\) have the same limit \(L\) then \(\lim_{n \to \infty} b_n = L\)

22. Example. Show \(\lim_{n \to \infty} \frac{\cos n}{n} = 0\)

Solution. Notice that for all \(n \geq 1\)

\[
0 \leq \left| \frac{\cos n}{n} \right| \leq \frac{1}{n}
\]

The result follows by applying the sandwich theorem.
23. Claim. The following are equivalent:
   (a) \( \lim_{n \to \infty} a_n = 0 \).
   (b) \( \lim_{n \to \infty} |a_n| = 0 \).

Proof. Exercise  
Hint: (Notice that \( d(|a_n|, 0) = d(a_n, 0) \))

24. Corollary. \( \lim_{n \to \infty} \frac{\cos n}{n} = 0 \)

25. Lemma. (Bernoull’s Inequality)  
If \( x > -1 \) then  
\[ (1 + x)^n \geq 1 + nx \]

Proof. By induction (Exercise).

26. Claim. If \( a > 1 \) then \( a^n \) is unbounded above.  
Proof. If \( a > 1 \) then \( a = 1 + x \) where \( x = a - 1 \). Notice \( x > 0 \). By Bernoulli’s inequality  
\[ a^n = (1 + x)^n > 1 + nx \]

Since \( \mathbb{N} \) is unbounded above, given \( M \) arbitrary, there exists \( N \in \mathbb{N} \) such that for all \( n > N, nx > M - 1 \). Hence, for all \( n > N \) \( a^n > M \) proving that the sequence \( (a^n) \) is unbounded above.

27. Claim. If \( 0 < a < 1 \) then \( \lim_{n \to \infty} a^n = 0 \)  
Proof. Since \( \frac{1}{a} > 1 \), it follows by the previous claim that the sequence \( \left( \frac{1}{a} \right)^n \) is unbounded. It follows that if \( \epsilon > 0 \) there exists \( N \) such that for all \( n > N \),  
\[ \left( \frac{1}{a} \right)^n > \frac{1}{\epsilon} \]. This implies that \( |a^n - 0| < \epsilon \) for all \( n > N \). Hence \( \lim_{n \to \infty} a^n = 0 \).

28. Definition. A sequence \( (x_n) \) in \( \mathbb{R} \) is:
   (a) \textit{increasing} if for all \( n, m \in \mathbb{N} \)  
   \[ n < m \Rightarrow x_n < x_m \]  
   (b) \textit{non-decreasing} if for all \( n, m \in \mathbb{N} \)  
   \[ n \leq m \Rightarrow x_n \leq x_m \]  
   (c) \textit{decreasing} if for all \( n, m \in \mathbb{N} \)  
   \[ n < m \Rightarrow x_n > x_m \]  
   (d) \textit{non-increasing} if for all \( n, m \in \mathbb{N} \)  
   \[ n \leq m \Rightarrow x_n \geq x_m \]

The following Venn diagram, expresses the relationships between these concepts:

29. Dedekind (Completeness) Axiom.  
Every non-empty subset of \( \mathbb{R} \) which is bounded above has a supremum (least upper bound).

30. Claim. A sequence \( \alpha = (a_n) \) of real numbers which is both non-decreasing bounded converges to the least upper bound of its set of terms.  
Proof. Let \( L = \sup \{a_n : n \in \mathbb{N} \} \) and \( \epsilon > 0 \). Then there exists \( N \) such that  
\[ L - \epsilon < a_N \leq L \]  
(otherwise \( L - \epsilon \) would be a smaller bound). Since \( \alpha \) is non-decreasing  
\[ n \geq N \Rightarrow L - \epsilon < a_n \leq L \]  
Hence \( \alpha \) converges to \( L \).
31. Corollary. A non-decreasing sequence \( \alpha = (a_n) \) of real numbers either converges or is unbounded above.

32. Definition. Let \((x_n)\) be a real sequence. A natural number \(m\) is called:
   (a) A \textit{peak point} of \((x_n)\) if \(x_m \geq x_n\) for all \(n \geq m\).
   (b) A \textit{\(p\)-blocker} if \(m > p\) and \(x_m \geq x_p\).

33. Bolzano Weierstrass Theorem [BW]. A bounded sequence has a convergent subsequence.
Proof. (See Spivak).
Let \(m_1 < m_2, \ldots, \) be the set of peak points. If this set is infinite then the bounded non increasing subsequence
\[ x_{m_1} \geq x_{m_2} \geq x_{m_k} \ldots \]
must converge (to its infimum). If, on the other hand, the set of peak points is finite, there exists a last peak point \(m\) in which case there exists a convergent subsequence of \(m\)-blockers defined recursively as follows: let \(q_1 = m + 1\) and, for each \(k\), let \(q_{k+1}\) be the minimum blocker of \(q_k\). Then the subsequence
\[ x_{q_1} \leq x_{q_2} \leq x_{q_k} \ldots \]
is non decreasing and bounded below so converges.

34. Exercise. In the proof of BW above, explain why the minimum at each stage exists.

35. Definition. A real sequence \((a_n)\) is \textit{Cauchy} if for every \(\epsilon > 0\) there exists \(N\) such that
\[ n, m > N \Rightarrow |a_n - a_m| < \epsilon \]

36. Claim. Cauchy sequences are bounded.
Proof. Let \(\epsilon > 0\). There exists \(N\) such that for \(m, n \geq N\), \(|a_m - a_n| < \epsilon\). Then, by the triangle inequality
\[ |a_m| - |a_n| \leq |a_m - a_n| < \epsilon \]
for all \(m, n \geq N\). Taking \(n = N\). Then
\[ |a_m| - |a_N| < \epsilon \]
for all \(m \geq N\). It follows easily that \((a_m)\) is bounded by
\[ \pm \max \{|a_0|, |a_1|, \ldots, |a_{N-1}|, |a_N|, \epsilon + |a_N|\} \]

37. Claim [Uses BW]. Every Cauchy sequence in \(\mathbb{R}\) converges.
Proof. Let \((a_n)\) be a Cauchy sequence of real numbers. Then \((a_n)\) is bounded so by BW has a convergent subsequence \((a_{n_k})\). Let \(L\) be the limit of this subsequence. Then there exist \(N\) such that
\[ n_k > N \Rightarrow |a_{n_k} - L| < \epsilon/2 \]
and there exists \(M\) such that
\[ n, n_k > M \Rightarrow |a_n - a_{n_k}| < \epsilon/2 \]
Then, for all \(n > \max\{N, M\}\)
\[ |a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \epsilon \]

38. Corollary. Dedekind completeness implies Cauchy completeness